DOI:
10.1016/j.aim.2019.01.025

ON THE ANALYSIS OF INCOMPLETE SPECTRA IN RANDOM MATRIX THEORY THROUGH AN EXTENSION OF THE JIMBO-MIWA-UENO DIFFERENTIAL

THOMAS BOTHNER, ALEXANDER ITS, AND ANDREI PROKHOROV

Abstract. Several distribution functions in the classical unitarily invariant matrix ensembles are prime examples of isomonodromic tau functions as introduced by Jimbo, Miwa and Ueno (JMU) in the early 1980s [50]. Recent advances in the theory of tau functions [47], based on earlier works of B. Malgrange and M. Bertola, have allowed to extend the original Jimbo-Miwa-Ueno differential form to a 1-form closed on the full space of extended monodromy data of the underlying Lax pairs. This in turn has yielded a novel approach for the asymptotic evaluation of isomonodromic tau functions, including the exact computation of all relevant constant factors. We use this method to efficiently compute the tail asymptotics of soft-edge, hard-edge and bulk scaled distribution and gap functions in the complex Wishart ensemble, provided each eigenvalue particle has been removed independently with probability $1-\gamma\in(0,1]$.

1. Introduction and statement of results

This paper is concerned with the large gap asymptotics of the universal limiting distributions in random matrix theory. The issue which we will specifically address is the evaluation of the constant factors appearing in these asymptotics, the so-called "constant problem". We will present a new method for the derivation of tail expansions which does not rely on Fredholm, or Toeplitz, or Hankel determinant formulæ, which are the usual tools in the analysis of distribution functions. Instead, our approach is based on the interpretation of the distribution functions as tau functions of the theory of isomonodromic deformations of certain systems of linear ODEs with rational coefficients. Specifically we shall evaluate, including the constant factors, the tail asymptotics of soft-edge, hard-edge and bulk scaled distribution and gap functions in the complex Wishart ensemble, provided each eigenvalue particle has been removed independently with probability $1-\gamma\in(0,1]$. In what follows, we shall describe the content of our work and its principal results in detail.

1.1. Complete Wishart ensemble.

The complex Wishart ensemble [57, 34] can be realized as a log-gas system of (eigenvalue) particles $0<\lambda_1<\lambda_2<\ldots<\lambda_n$ on the positive real axis with probability density function for the location of the λ_j's given by

$$f(\lambda_1,\ldots,\lambda_n)=\frac{1}{Z_n}\prod_{1\leq j<k\leq n}(\lambda_k-\lambda_j)^2\prod_{j=1}^n\lambda_j^{\alpha}\mathrm{e}^{-\lambda_j},\quad \alpha>-1. \tag{1.1}$$

The constant Z_n serves as normalization and (1.1) is also well-known under the name Laguerre Unitary Ensemble (LUE). The great benefit and applicability of a unitarily invariant ensemble (such as the LUE) stems from the fact that the point process (1.1) is determinantal, i.e. the underlying rescaled marginal densities (a.k.a. k-point correlation functions) can be computed in closed determinantal form, cf. [57, 34],

$$R_k(\lambda_1,\ldots,\lambda_k)=\frac{n!}{(n-k)!}\int_0^\infty\cdots\int_0^\infty f(\lambda_1,\ldots,\lambda_n)\prod_{j=k+1}^n\mathrm{d}\lambda_j=\det\left[K_n(\lambda_j,\lambda_\ell)\right]_{j,\ell=1}^k,\quad k=1,\ldots,n. \tag{1.2}$$

Here, the kernel function $K_n(x,y)$ is a Christoffel-Darboux kernel expressed in terms of classical Laguerre polynomials. It is well-known that (1.2) encodes the core integrable structure of the LUE and at the same

Date: January 5, 2019.

2010 *Mathematics Subject Classification.* Primary 60B20; Secondary 45M05, 82B26, 70S05, 33C10, 33C15.

Key words and phrases. Thinned LUE process, isomonodromic tau-functions, tail asymptotics, action integrals, Weibull statistics, Poisson statistics, Riemann-Hilbert problem, Deift-Zhou nonlinear steepest descent method.

T.B. acknowledges support of the AMS and the Simons Foundation through a travel grant as well as the hospitality of Indiana University-Purdue University Indianapolis in August 2016. A.I and A.P. acknowledge support by the NSF Grant DMS-1361856, the NSF Grant DMS-1700261 and the RSF grant #17-11-01126.

time paths the way to a rigorous analysis of the thermodynamical limit $n \to \infty$ of the k-point correlation function. Indeed, cf. [33, 34], the eigenvalue density obeys the Marchenko-Pastur law,

$$\lim_{n\to\infty} R_1(n\lambda) = \rho_{\mathrm{MP}}(\lambda) \equiv \frac{1}{2\pi}\sqrt{\frac{4-\lambda}{\lambda}}\chi_{(0,4)}(\lambda), \qquad \chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}, \tag{1.3}$$

shown in Figure 1 below. The global limiting law (1.3) leads in turn to three qualitatively different local scenarios: provided we center and scale correctly,

$$\begin{cases} \mu_j^S &= 2^{-\frac{4}{3}} n^{-\frac{1}{3}}(\lambda_j - 4n) \quad \text{soft-edge} \\ \mu_j^H &= 4n\lambda_j \quad \text{hard-edge}, \qquad c \in (0,4) \ \ \text{fixed}; \qquad j = 1,\ldots,n, \\ \mu_j^B &= \rho_{\mathrm{MP}}(cn)(\lambda_j - cn) \quad \text{bulk} \end{cases}$$

then, cf. [34], for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\big(\#\{\mu_j^S \in (t,+\infty)\} = 0\big) = F_S(t) \tag{1.4}$$

which is the limiting distribution function of the largest eigenvalue in the LUE, and for $t \in \mathbb{R}_{>0}$,

$$\lim_{n\to\infty}\Big(1 - \mathbb{P}\big(\#\{\mu_j^H \in (0,t)\} = 0\big)\Big) = 1 - F_H(t,\alpha); \quad \text{resp.} \quad \lim_{n\to\infty}\mathbb{P}\left(\#\left\{\mu_j^B \in \left(-\frac{t}{\pi},\frac{t}{\pi}\right)\right\} = 0\right) = F_B(t) \tag{1.5}$$

which is the limiting distribution function of the smallest eigenvalue, resp. the limiting bulk gap function in the LUE.

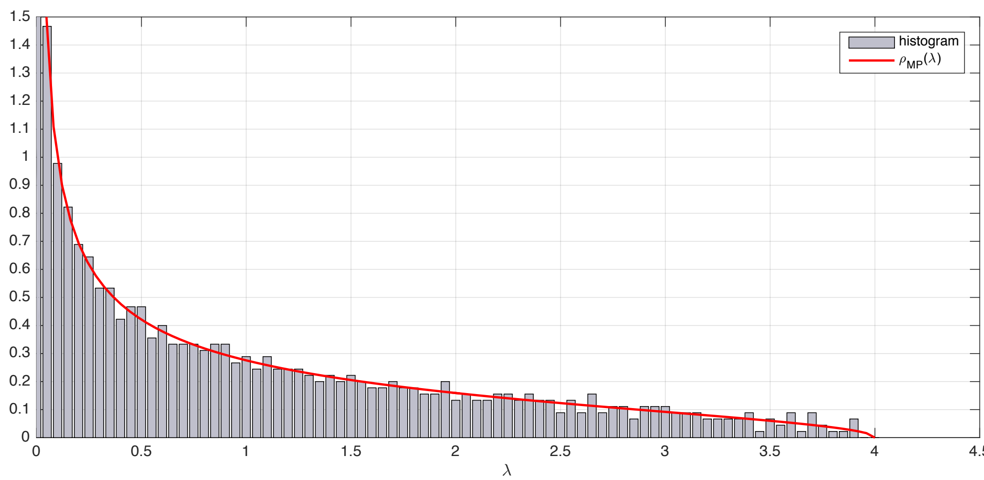

FIGURE 1. Histogram of the eigenvalues of a 900×900 (rescaled) complex Wishart matrix in comparison with the Marchenko-Pastur density (1.3).

The intimate connection of the three functions $F_B(t), F_S(t)$ and $F_H(t,\alpha)$ defined in (1.4) and (1.5) to the theory of integrable systems is remarkable and well-known: first, for the bulk function, as proven by Jimbo-Miwa-Mori-Sato [49],

$$\ln F_B(t) = \int_0^t \mathcal{H}_B\big(q(s),p(s),s\big)\,\mathrm{d}s, \qquad t \in \mathbb{R}_{\geq 0} \tag{1.6}$$

in terms of the Hamiltonian dynamical system

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial \mathcal{H}_B}{\partial q}, \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial \mathcal{H}_B}{\partial p}; \qquad \mathcal{H}_B(q,p,t) = -4\mathrm{i}q + \frac{4}{t}q^2\sinh^2\left(\frac{p}{2}\right). \tag{1.7}$$

The required solutions (q,p) to this system are smooth on the positive real axis and uniquely determined by the boundary behavior $q(t)\sim\frac{1}{2\pi\mathrm{i}}$ and $p(t)\sim 4it$ as $t\downarrow 0$. The dynamical system (1.7) is equivalent to a special case of the Painlevé-V equation for the function $\omega(t)=\exp(p(\frac{t}{2}))$, cf. [49],

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}t^2}=\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)^2\frac{3\omega-1}{2\omega(\omega-1)}+\frac{2\omega(\omega+1)}{\omega-1}+\frac{2i\omega}{t}-\frac{1}{t}\frac{d\omega}{dt}\tag{1.8}$$

Second, for the distribution function of the largest eigenvalue, as proven by Tracy-Widom [64],

$$\ln F_S(t)=-\int_t^\infty\mathcal{H}_S\big(q(s),p(s),s\big)\,\mathrm{d}s,\qquad t\in\mathbb{R}\tag{1.9}$$

in terms of the Hamiltonian dynamical system

$$\frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\partial\mathcal{H}_S}{\partial q},\quad\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\partial\mathcal{H}_S}{\partial p};\qquad\qquad\mathcal{H}_S(q,p,t)=\frac{1}{4}p^2-tq^2-q^4.\tag{1.10}$$

Here the solutions (q,p) are smooth on the real line and fixed in such a way that $q(t)\sim\mathrm{Ai}(t)$ and $p(t)=2q_t(t)$ as $t\to+\infty$, where $\mathrm{Ai}(z)$ is the Airy function, cf. [60]. The system (1.10) is equivalent to a special case of the Painlevé II equation for the function $q(t)$,

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2}=tq+2q^3,\tag{1.11}$$

and the solution $q(t)$ selected by the condition $q(t)\sim\mathrm{Ai}(t)$ is known as the Hastings-McLeod solution to (1.11), see [40]. Third, again by Tracy-Widom [65],

$$\ln F_H(t,\alpha)=\int_0^t\mathcal{H}_H\big(q(s,\alpha),p(s,\alpha),s,\alpha\big)\,\mathrm{d}s,\qquad t\in\mathbb{R}_{\geq0},\quad\alpha\in\mathbb{R}_{>-1}\tag{1.12}$$

in terms of the Hamiltonian dynamical system

$$\frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\partial\mathcal{H}_H}{\partial q},\quad\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\partial\mathcal{H}_H}{\partial p};\qquad\qquad\mathcal{H}_H(q,p,t,\alpha)=\frac{q^2-1}{4t}p^2-\frac{\alpha^2q^2}{4t(q^2-1)}-\frac{q^2}{4}.\tag{1.13}$$

We enforce

$$q(t,\alpha)\sim\frac{t^{\frac{1}{2}\alpha}}{2^\alpha\Gamma(1+\alpha)},\qquad p(t,\alpha)=\frac{2tq_t(t,\alpha)}{q^2(t,\alpha)-1},\quad t\downarrow0,\qquad q_t=\frac{\mathrm{d}q}{\mathrm{d}t},$$

where $\Gamma(z)$ is Euler's Gamma function. In addition, $q^2(t,\alpha)$ is smooth and real-valued on the half ray $(0,+\infty)\subset\mathbb{R}$. Dynamical system (1.13) is equivalent to yet another special case of the Painlevé-V equation for the function $y(t)=(q(t^2)-1)/(q(t^2)+1)$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\frac{3y-1}{2y(y-1)}-\frac{2y(y+1)}{y-1}-\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\alpha^2}{8}\frac{(y-1)^2}{t^2}\left(y-\frac{1}{y}\right).\tag{1.14}$$

Remark 1.1. *The aforementioned smoothness properties of (q,p) in (1.7) and (1.10) are well-known, cf. [24, 40]. The smoothness of q and p in the case (1.13) is proven in Appendix C.*

Remark 1.2. *Each Hamiltonian $\mathcal{H}$ listed above solves itself a σ-Painlevé equation in the variable t, see for instance [34], Chapter 8.*

From (1.6), (1.9) and (1.12) we see that $F_B(t)$, $F_S(t)$ and $F_H(t,\alpha)$ are generating functions of Hamiltonians associated with specific Painlevé systems. As such they are directly related to the theory of isomonodromic tau-functions in the sense of Jimbo-Miwa-Ueno [50]. We will discuss this connection in more detail in Section 3 below.

1.2. **Incomplete Wishart ensemble.** We now return to the discussion of the complex Wishart ensemble and the collection of soft-edge, hard-edge and bulk scaled eigenvalues $\{\mu_j^S,\mu_j^H,\mu_j^B\}_{j=1}^n$. But instead of the complete setup (1.1) we will be interested in the following thinned/incomplete Wishart ensemble (cf. [9, 10, 11]): fix $\gamma\in[0,1]$ and discard each (either soft-edge, or hard-edge or bulk scaled) eigenvalue $\mu_j^r, r=S,H,B$ independently with probability $1-\gamma$. This operation reduces correlation in our initial setup and introduces a new particle system on the real line,

$$\mu_{1,\gamma}^r<\mu_{2,\gamma}^r<\ldots<\mu_{N,\gamma}^r\qquad r=S,H,B;\qquad N=N(n,\gamma)\leq n.$$

Quite naturally we are interested in the statistical properties of this new system, in particular what can be said about the thinned extremal distributions

$$\lim_{n\to\infty}\mathbb{P}\big(\#\{\mu_{j,\gamma}^S\in(t,+\infty)\}=0\big)=F_S(t;\gamma);\qquad \lim_{n\to\infty}\Big(1-\mathbb{P}\big(\#\{\mu_{j,\gamma}^H\in(0,t)\}=0\big)\Big)=1-F_H(t,\alpha;\gamma),\tag{1.15}$$

and the thinned bulk gap function

$$\lim_{n\to\infty}\mathbb{P}\left(\#\left\{\mu_{j,\gamma}^B\in\left(-\frac{t}{\pi},\frac{t}{\pi}\right)\right\}=0\right)=F_B(t;\gamma),\tag{1.16}$$

which directly generalize (1.4) and (1.5)? As it turns out the thinning mechanism preserves Hamiltonian structure as summarized in our first result below.

Theorem 1.3. *Fix $\gamma\in[0,1]$ and let $F_B(t;\gamma),F_S(t;\gamma)$ and $F_H(t,\alpha;\gamma)$ denote the functions defined in (1.15) and (1.16). Then*

$$\ln F_B(t;\gamma)=\int_0^t\mathcal{H}_B\big(q(s;\gamma),p(s;\gamma),s\big)\,\mathrm{d}s,\qquad \ln F_S(t;\gamma)=-\int_t^\infty\mathcal{H}_S\big(q(s;\gamma),p(s;\gamma),s\big)\,\mathrm{d}s,$$

and

$$\ln F_H(t,\alpha;\gamma)=\int_0^t\mathcal{H}_H\big(q(s,\alpha;\gamma),p(s,\alpha;\gamma),s,\alpha\big)\,\mathrm{d}s,$$

using the Hamiltonians in (1.7), (1.10), (1.13) and the solutions to the underlying dynamical systems are fixed as follows: for the bulk function after thinning,

$$q(t;\gamma)\sim\frac{\gamma}{2\pi\mathrm{i}},\quad p(t;\gamma)\sim4\mathrm{i}t,\qquad t\downarrow0;\tag{1.17}$$

whereas for the distribution function of the largest eigenvalue after thinning the Hastings-McLeod solution is replaced by the Ablowitz-Segur solution [1], i.e.

$$q(t;\gamma)\sim\sqrt{\gamma}\,\mathrm{Ai}(t),\quad p(t;\gamma)=2q_t(t,\gamma),\qquad t\to+\infty.\tag{1.18}$$

In addition, related to the distribution function of the smallest eigenvalue after thinning,

$$q(t,\alpha;\gamma)\sim\frac{\sqrt{\gamma}\,t^{\frac12\alpha}}{2^\alpha\Gamma(1+\alpha)},\quad p(t,\alpha;\gamma)=\frac{2tq_t(t,\alpha;\gamma)}{q^2(t,\alpha;\gamma)-1},\qquad t\downarrow0.\tag{1.19}$$

The proof of Theorem 1.3 follows from a combination of standard arguments based on the Fredholm determinant representations of $F_B(t),F_S(t)$ and $F_H(t,\alpha)$, see Section 2 below. In order to prepare for our next objective we remind the reader that the thinning process weakens correlations from the initial setup $\{\mu_j^S,\mu_j^H,\mu_j^B\}_{j=1}^n$, thus varying γ we are able to interpolate between particle systems that obey random matrix theory statistics and systems modeled by more classical distribution families, e.g. Poisson and Weibull, see Subsection 1.4 below. This interpolation mechanism is well-known by now, see e.g. [9, 11, 16, 17] and the analytic challenge lies in the derivation of tail expansions for $F_B(t;\gamma),F_S(t;\gamma)$ and $F_H(t,\alpha;\gamma)$ as $t\downarrow0,t\to+\infty$ (bulk), $t\to\pm\infty$ (soft-edge) and $t\downarrow0,t\to+\infty$ (hard-edge) which are uniform with respect to $\gamma\in[0,1]$.

Remark 1.4. *The uniformity requirement poses a clear challenge: the introduction of γ into the boundary conditions in Theorem 1.3 has a very subtle effect on, both, analytic and asymptotic properties of (q,p). For instance, in case of (1.18), solutions are bounded on the entire real axis for $\gamma\in[0,1)$, but unbounded (as $t\to-\infty$) once $\gamma=1$. Similar phenomena also occur for (1.17) and (1.19) and we shall return to these interesting phase transitions after the next two subsections.*

1.3. **Tail asymptotics and action integral formulæ.** The principal analytical question concerning the distribution functions $F_B(t;\gamma)$, $F_H(t,\alpha;\gamma)$ and of $F_S(t;\gamma)$ is their *tail asymptotics*, i.e., the behavior of $F_B(t;\gamma), F_H(t,\alpha;\gamma)$ as $t\downarrow 0$ or $t\to\infty$, and of $F_S(t;\gamma)$ as $t\to\pm\infty$. In view of Theorem 1.3 we realize at once that half of the tail expansions are easy to compute. Indeed upon substitution of the boundary data (1.17), (1.18), (1.19) into the Hamiltonian formulæ we obtain immediately the leading order behavior of $F_B(t;\gamma), F_H(t,\alpha;\gamma)$ as $t\downarrow 0$ and of $F_S(t;\gamma)$ as $t\to+\infty$,

$$F_B(t;\gamma)=1-\frac{2\gamma}{\pi}t\big(1+o(1)\big),\quad t\downarrow 0;\qquad F_S(t;\gamma)=1-\frac{\gamma}{16\pi}t^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}t^{\frac{3}{2}}}\big(1+o(1)\big),\quad t\to+\infty;$$

and

$$F_H(t,\alpha;\gamma)=1-\frac{\gamma}{\Gamma^2(2+\alpha)}\left(\frac{t}{4}\right)^{\alpha+1}\big(1+o(1)\big),\quad t\downarrow 0.$$

Most importantly, these expansions are uniform with respect to $\gamma\in[0,1]$ and (in case of $F_H(t,\alpha;\gamma)$) α chosen from compact subsets of $(-1,+\infty)\subset\mathbb{R}$. Much more challenging are the remaining three tails: for these we could in principle use Painlevé asymptotic information, see [60], Chapter 32. For instance in case of (1.10) it is known that

$$q(t;\gamma)=(-t)^{-\frac{1}{4}}\sqrt{\frac{v}{\pi}}\cos\left(\frac{2}{3}(-t)^{\frac{3}{2}}-\frac{v}{2\pi}\ln\big(8(-t)^{\frac{3}{2}}\big)+\phi\right)+\mathcal{O}\left(t^{-1}\right),\quad t\to-\infty,$$

where $v=-\ln(1-\gamma), \phi=\frac{\pi}{4}-\arg\Gamma(\frac{v}{2\pi\mathrm{i}})$ and $\gamma\in[0,1)$ is fixed. In addition,

$$q(t;1)=\sqrt{-\frac{t}{2}}\left(1+\frac{1}{8t^3}+\mathcal{O}\left(t^{-6}\right)\right),\quad t\to-\infty.$$

Thus, upon t-differentiation of the Hamiltonian formula in Theorem 1.3 and subsequent indefinite integration,

$$\ln F_S(t;\gamma)=-\frac{2v}{3\pi}(-t)^{\frac{3}{2}}+\frac{v^2}{4\pi^2}\ln\big(8(-t)^{\frac{3}{2}}\big)+\mathcal{O}(1),\quad t\to-\infty,\quad \gamma\in[0,1); \tag{1.20}$$

as well as

$$\ln F_S(t;1)=\frac{t^3}{12}-\frac{1}{8}\ln(-t)+\mathcal{O}(1),\quad t\to-\infty. \tag{1.21}$$

A similar approach can be carried out for (1.7) and (1.13) once we use the relevant asymptotic information given in [62, 58, 59, 2]. But in either case, the outlined method does not allow us to compute the $\mathcal{O}(1)$ terms (see (1.20) and (1.21)) in an efficient way. And these terms are needed for the rigorous analysis of the phase transition as $\gamma\uparrow 1$, for instance $\mathcal{O}(1)$ in (1.20) is bounded as $t\to-\infty$ but not as $\gamma\uparrow 1$. The problem of finding these terms is sometime referred to as the "constant problem" and this is the main issue we are addressing in this paper.

The usual approaches to the computation of the above mentioned outstanding constant factors, as well as the description of the full transitional regime, are based on the utilization of additional integrable structures, i.e., Fredholm determinant formulæ (see Section 2 below),

$$F_B(t;\gamma)=\det\big(1-\gamma K_{\sin}\restriction_{L^2(-t,t)}\big),\qquad F_S(t;\gamma)=\det\big(1-\gamma K_{\mathrm{Ai}}\restriction_{L^2(t,\infty)}\big), \tag{1.22}$$

and

$$F_H(t,\alpha;\gamma)=\det\big(1-\gamma K_{\mathrm{Bess}}^{\alpha}\restriction_{L^2(0,t)}\big). \tag{1.23}$$

Here, $K_{\sin}, K_{\mathrm{Ai}}$ and $K_{\mathrm{Bess}}^{\alpha}$ are the trace-class integral operators with kernels

$$K_{\sin}(\lambda,\mu)=\frac{\sin(\lambda-\mu)}{\pi(\lambda-\mu)},\qquad K_{\mathrm{Ai}}(\lambda,\mu)=\int_0^\infty \mathrm{Ai}(\lambda+s)\mathrm{Ai}(\mu+s)\,\mathrm{d}s, \tag{1.24}$$

and

$$K_{\mathrm{Bess}}^{\alpha}(\lambda,\mu)=\frac{1}{4}\int_0^1 J_\alpha(\sqrt{\lambda s})J_\alpha(\sqrt{\mu s})\,\mathrm{d}s, \tag{1.25}$$

where $J_\alpha(z)$ is the Bessel function of order α. Based on these formulæ one can now either use discretization techniques (e.g. representing $F_B(t;\gamma)$ as limit of a Toeplitz determinant [28, 52, 23], or $F_S(t;\gamma)$ as Hankel determinant limit [22, 53]), or apply operator theoretical arguments [5, 20, 29, 30, 66], or refer to the algebra of integrable operators [45, 24]. With these tools at hand, it is possible to improve (1.20) and (1.21) (see [16, 22, 3]),

$$\ln F_S(t;\gamma) = -\frac{2v}{3\pi}(-t)^{\frac{3}{2}} + \frac{v^2}{4\pi^2}\ln\big(8(-t)^{\frac{3}{2}}\big) + \ln\left(G\left(1+\frac{\mathrm{i}v}{2\pi}\right)G\left(1-\frac{\mathrm{i}v}{2\pi}\right)\right) + o(1), \quad t\to-\infty,\ \gamma\in[0,1);$$

and

$$\ln F_S(t;1) = \frac{t^3}{12} - \frac{1}{8}\ln(-t) + \zeta'(-1) + \frac{1}{24}\ln 2 + o(1), \quad t\to-\infty$$

in terms of the Barnes G-function $G(z)$ and the Riemann zeta-function $\zeta(z)$.

In this paper we present a new method for the derivation of tail expansions which does not rely on Fredholm, or Toeplitz, or Hankel determinant formulæ but instead on the Hamiltonian system approach (see Theorem 1.3) to the gap and distribution functions. In this approach the already available Painlevé asymptotic information (compare derivation of (1.20) and (1.21)) will be sufficient to obtain full leading order asymptotic information for $F_B(t;\gamma), F_S(t;\gamma)$ and $F_H(t,\alpha;\gamma)$ provided $\gamma\in[0,1)$ is fixed. Our method is based on the following action integral formulæ which form our second result.

Theorem 1.5. *Let $F_B(t;\gamma), F_S(t;\gamma)$ and $F_H(t,\alpha;\gamma)$ be defined as in (1.15), (1.16) for $\gamma\in[0,1]$ and (q,p) specified as in Theorem 1.3. Then*

$$\ln F_B(t;\gamma) = t\,\mathcal{H}_B(q,p,t) - pq + I_B(t;\gamma), \qquad I_B(t;\gamma) = \int_0^t \big(pq_s - \mathcal{H}_B(q,p,s)\big)\,\mathrm{d}s, \tag{1.26}$$

and

$$\ln F_S(t;\gamma) = \frac{1}{3}\big(2t\,\mathcal{H}_S(q,p,t) - pq\big) + I_S(t;\gamma), \qquad I_S(t;\gamma) = -\int_t^\infty \big(pq_s - \mathcal{H}_S(q,p,s)\big)\,\mathrm{d}s \tag{1.27}$$

where the integration path in the action integrals I_B and I_S is chosen on the real line. In addition,

$$\ln F_H(t,\alpha;\gamma) = 2t\,\mathcal{H}_H(q,p,t,\alpha) - L(t,\alpha;\gamma) + I_H(t,\alpha;\gamma), \tag{1.28}$$

with

$$L(t,\alpha;\gamma) = \frac{\alpha^2}{2}\int_0^t \frac{q^2\,\mathrm{d}s}{s(q^2-1)}, \qquad I_H(t,\alpha;\gamma) = \int_0^t \big(pq_s - \mathcal{H}_H(q,p,s,\alpha)\big)\,\mathrm{d}s, \qquad \alpha\geq 0;$$

and

$$L(t,\alpha;\gamma) = \frac{\alpha^2}{2}\int_0^t \frac{\mathrm{d}s}{s(q^2-1)}, \qquad I_H(t,\alpha;\gamma) = \int_0^t \left(pq_s - \frac{\alpha^2}{2s} - \mathcal{H}_H(q,p,s,\alpha)\right)\mathrm{d}s, \quad -1<\alpha<0.$$

The integration paths for L and I_H lie in the half-plane $\Re s>0$ and avoid the discrete set $\{s\in\mathbb{C}: q^2(s,\alpha;\gamma)=1\}$.

In order to appreciate the usefulness of this theorem for the evaluation of constant factors in tail asymptotics, let us highlight the difficulties which one faces in the existing approaches to the problem. We will restrict ourselves to the sine - kernel distribution F_B and consider the asymptotic scheme based on the theory of integrable Fredholm operators, cf. [45, 24].

We shall start with the classical differential identity which is the beginning of almost every study of Fredholm determinants,

$$\frac{\partial}{\partial\gamma}\ln F_B(t;\gamma) = \frac{\partial}{\partial\gamma}\ln\det\big(1-\gamma K_{\sin}\restriction_{L^2(-t,t)}\big) = -\frac{1}{\gamma}\mathrm{Trace}\,R_B = -\frac{1}{\gamma}\int_{-t}^t R_B(\lambda,\lambda;t,\gamma)d\lambda. \tag{1.29}$$

Here, R_B is the resolvent of the operator $K_{\sin}$ defined by the usual formula, $R_B = \gamma\big(1-\gamma K_{\sin}\big)^{-1}K_{\sin}$ and $R_B(\lambda,\mu;t,\gamma)$ is its kernel. As the sine-kernel belongs to the class of integrable Fredholm operators (see, e.g., [24]), the resolvent kernel $R_B(\lambda,\mu;t,\gamma)$ admits the following explicit representation in terms of the 2×2 matrix valued solution $\mathbf{Y}(\lambda)\equiv\mathbf{Y}(\lambda;t,\gamma)$ to a Riemann-Hilbert problem (see RH problem (5.1) in Section 5 for more detail),

$$R(\lambda,\mu) = \frac{1}{2\pi i(\lambda-\mu)}\begin{bmatrix}\mathrm{e}^{-\mathrm{i}\mu} & -\mathrm{e}^{\mathrm{i}\mu}\end{bmatrix}\mathbf{Y}^{-1}\left(\frac{\mu}{t}\right)\mathbf{Y}\left(\frac{\lambda}{t}\right)\begin{bmatrix}\mathrm{e}^{\mathrm{i}\lambda}\\ \mathrm{e}^{-\mathrm{i}\lambda}\end{bmatrix} \tag{1.30}$$

Therefore, if one knew the large t-asymptotics of the solution $\mathbf{Y}(\lambda; t, \gamma)$ to RHP 5.1 which are uniform with respect to γ and λ, the needed large t asymptotics of F_B *including the constant term* could have been determined via double integration,

$$\ln F_B(t, \gamma) = -\int_0^\gamma \int_{-t}^t R_B(\lambda, \lambda; t; \gamma)\, \mathrm{d}\lambda \frac{\mathrm{d}\gamma}{\gamma}. \tag{1.31}$$

In case $\gamma \in [0, 1)$ all needed asymptotic information about $\mathbf{Y}(\lambda; t, \gamma)$ can indeed be extracted via the non-linear steepest descent analysis of Riemann-Hilbert problem 5.1 ([17]; see also Section 5 where this analysis is reproduced). However, the relevant formulæ, though explicit, are very complicated. The asymptotic evaluation of the double integral in (1.31) becomes enormously difficult, and in fact has never been done. The main difficulty lies in the non-locality of the differential identity (1.29) and, as a consequence, of the integral formula (1.31) in the variable λ. There is though a way to circumvent this double integration, and it uses the already mentioned isomonodromy connection of the distribution function F_B. Indeed, with respect to λ, the matrix function $\mathbf{Y}(\lambda)$ satisfies a linear differential equation with rational coefficients (see [24]) which allows one to evaluate the most challenging integral in (1.31) - namely, the integral in λ, and replace the differential identity (1.29) by the following formula (cf. [19]),

$$\frac{\partial}{\partial\gamma} \ln F_B(t; \gamma) = \frac{\partial}{\partial\gamma} \ln\det(1 - \gamma K_{\sin}\restriction_{L^2(-t,t)}) = -2\mathrm{i}t \frac{\partial}{\partial\gamma} Y_1^{11} + \frac{\gamma}{2\pi\mathrm{i}} \mathrm{trace}\left\{ \widehat{\mathbf{Y}}^{-1}(1) \frac{\partial}{\partial\gamma} \widehat{\mathbf{Y}}(1) \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$$
$$- \frac{\gamma}{2\pi\mathrm{i}} \mathrm{trace}\left\{ \widehat{\mathbf{Y}}^{-1}(-1) \frac{\partial}{\partial\gamma} \widehat{\mathbf{Y}}(-1) \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}, \tag{1.32}$$

where Y_1^{11} and $\widehat{\mathbf{Y}}(\lambda)$ are defined at the beginning of Section 5 - see formulation of RHP 5.1, properties (3) and (4). This relation is still not very simple, but it involves only *local* characteristica of the solution $\mathbf{Y}(\lambda)$. This locality allows one to use (1.32) for evaluating the large t asymptotics of F_B, although the calculations which one has to go through are still very tough.

The value of Theorem 1.5 lies in the fact that it yields an alternative to (1.32) local γ - differential formula for F_B, as well as similar formulæ for the other two distribution functions, which would simplify dramatically their asymptotic analysis. Indeed, identities (1.26) - (1.28) transform the original Hamiltonian integrals of Theorem 1.3 to the *action integrals* I_r plus explicit terms. The latter are either already localized, i.e. without any integrals, or integral terms as in (1.28) that admit a straightforward Riemann-Hilbert representation (see Section 6 below). The great advantage of having the full classical action integral instead of its truncated form lies in the fundamental fact that the variational derivatives of the classical action and, in particular, the γ - derivatives of the classical actions $I_{B,S,H}$ are simple local functions of the canonical variables p and q. For instance, for the integral I_B, we would have,

$$\frac{\partial I_B}{\partial\gamma} = \int_0^t \left(q_s p_\gamma + p(q_\gamma)_s - \frac{\partial\mathcal{H}_B}{\partial p} p_\gamma - \frac{\partial\mathcal{H}_B}{\partial q} q_\gamma \right) \mathrm{d}s; \quad \text{where } f_\gamma = \frac{\partial f}{\partial\gamma},$$

and integrating by parts the second term,

$$\frac{\partial I_B}{\partial\gamma} = pq_\gamma\Big|_{s=0}^t + \int_0^t \left(q_s p_\gamma - p_s q_\gamma - \frac{\partial\mathcal{H}_B}{\partial p} p_\gamma - \frac{\partial\mathcal{H}_B}{\partial q} q_\gamma \right) \mathrm{d}s,$$
$$= pq_\gamma\Big|_{s=0}^t + \int_0^t \left(\left(q_s - \frac{\partial\mathcal{H}_B}{\partial p} \right) p_\gamma - \left(p_s + \frac{\partial\mathcal{H}_B}{\partial q} \right) q_\gamma \right) \mathrm{d}s = pq_\gamma$$

where the remaining integral term vanishes due to the dynamical equations (1.7) and $pq_\gamma|_{s=0} = 0$ because of the boundary behavior of $q(t)$ and $p(t)$ at $t = 0$. Similar calculations can be done for the other two action integrals and we arrive at the following important formulaæ.

Proposition 1.6. *With (q, p) as in Theorem 1.3,*

$$\frac{\partial I_r}{\partial\gamma} = pq_\gamma \quad r = B, S; \qquad \frac{\partial I_H}{\partial\gamma} = \begin{cases} pq_\gamma, & \alpha \geq 0 \\ pq_\gamma - \frac{\alpha}{2\gamma}, & -1 < \alpha < 0 \end{cases}; \qquad f_\gamma = \frac{\partial f}{\partial\gamma}.$$

Moreover,

$$\frac{\partial I_H}{\partial\alpha}=\begin{cases}pq_\alpha+\frac{1}{\alpha}L(t;\alpha,\gamma), & \alpha>0\\ pq_\alpha-\frac{\alpha}{2}\ln t+\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\big(2^\alpha\Gamma(1+\alpha)\big)+\frac{1}{\alpha}L(t,\alpha;\gamma), & -1<\alpha<0\end{cases};\qquad f_\alpha=\frac{\partial f}{\partial\alpha}.$$

One clearly notices how much simpler the differential formulæ of this Proposition are than identity (1.32).

The formal proof of Theorem 1.5 is easy, and it can be obtained through t-differentiation of both sides of equations (1.26), (1.27) and (1.28) with the simultaneous use of the respective Hamiltonian systems. This formal proof will be presented in Section 2. The methodological deficiency of this proof is that it does not provide any clue on where equations (1.26), (1.27) and (1.28) came from. In Section 3 we outline an alternative proof of these formulæ which, simultaneously, reveals their theoretical origin. This alternative proof is based on the tau function interpretation of the gap/distribution functions $F_{B,S,H}$, and it uses the extension of the Jimbo-Miwa-Ueno tau function differential to a differential 1-form whose external derivative coincides with the corresponding symplectic form. Hence the connection of the distribution functions in question to the relevant action integrals is not an accident; in fact, it is their *intrinsic property*.

1.4. **Large gap expansions and phase transitions.** As mentioned above, a direct application of (1.26), (1.27) and (1.28) is provided with the efficient and quick derivation of tail expansions for all three functions $F_B(t;\gamma),F_S(t;\gamma),F_H(t,\alpha;\gamma)$ in case $\gamma\in[0,1)$ and $\alpha>-1$ are fixed. Our third result is as follows.

Theorem 1.7. *For any fixed $\gamma\in[0,1),\alpha>-1$ there exist positive constants $t_0=t_0(\gamma,\alpha)$ and $c_r=c_r(\alpha,\gamma),r=B,S,H$ such that*

$$\ln F_B(t;\gamma)=-\frac{2v}{\pi}t+\frac{v^2}{2\pi^2}\ln(4t)+2\ln\left(G\left(1+\frac{\mathrm{i}v}{2\pi}\right)G\left(1-\frac{\mathrm{i}v}{2\pi}\right)\right)+r_B(t;\gamma)\quad\forall\,t\geq t_0,\tag{1.33}$$

followed by

$$\ln F_S(t;\gamma)=-\frac{2v}{3\pi}|t|^{\frac32}+\frac{v^2}{4\pi^2}\ln\left(8|t|^{\frac32}\right)+\ln\left(G\left(1+\frac{\mathrm{i}v}{2\pi}\right)G\left(1-\frac{\mathrm{i}v}{2\pi}\right)\right)+r_S(t;\gamma)\quad\forall\,(-t)\geq t_0,\tag{1.34}$$

and concluding with

$$\ln F_H(t,\alpha;\gamma)=-\frac{v}{\pi}\sqrt{t}+\frac{v^2}{8\pi^2}\ln(16t)+\frac{\alpha}{2}v+\ln\left(G\left(1+\frac{\mathrm{i}v}{2\pi}\right)G\left(1-\frac{\mathrm{i}v}{2\pi}\right)\right)+r_H(t,\alpha;\gamma)\quad\forall\,t\geq t_0.\tag{1.35}$$

Here, $v=-\ln(1-\gamma)\in[0,+\infty)$, $G(z)$ is the Barnes G-function and the t-differentiable error terms satisfy

$$\big|r_B(t;\gamma)\big|\leq\frac{c_B(\gamma)}{t}\quad\forall\,t\geq t_0;\qquad\big|r_S(t;\gamma)\big|\leq\frac{c_S(\gamma)}{|t|^{\frac34}}\quad\forall\,(-t)\geq t_0;\qquad\big|r_H(t,\alpha;\gamma)\big|\leq\frac{c_H(\alpha;\gamma)}{\sqrt{t}}\quad\forall\,t\geq t_0.$$

Remark 1.8. *Expansion (1.33) was first derived in [20] with the indicated error estimate given in [17]. This expansion also follows from the general results of [5]. The first proof of (1.34) appeared recently in [16], resolving an earlier conjecture posed in [8]. Expansion (1.35), to the best of our knowledge, is completely new.*

Expansions (1.33), (1.34) and (1.35) are valid provided each particle $\mu_j^r,r=S,H,B$ has been removed with positive probability $1-\gamma\in(0,1]$. As such they are in sharp contrast to the following three expansions (compare e.g. (1.21) above), [29, 52, 23, 28]

$$\ln F_B(t;1)=-\frac{t^2}{2}-\frac14\ln t+3\zeta'(-1)+\frac{1}{12}\ln 2+o(1),\quad t\to+\infty;\tag{1.36}$$

and [22, 3]

$$\ln F_S(t;1)=\frac{t^3}{12}-\frac18\ln(-t)+\zeta'(-1)+\frac{1}{24}\ln 2+o(1),\quad t\to-\infty;\tag{1.37}$$

as well as [30, 25]

$$\ln F_H(t,\alpha;1)=-\frac{t}{4}+\alpha\sqrt{t}-\frac{\alpha^2}{4}\ln t+\ln\left(\frac{G(1+\alpha)}{(2\pi)^{\frac{\alpha}{2}}}\right)+o(1),\quad t\to+\infty.\tag{1.38}$$

In each case the gap or distribution function approaches zero (or one in case of $1-F_H$) faster in the complete setup opposed to thinned version thereof. Thus a non-trivial phase transition occurs as $\gamma\uparrow 1$. In fact, using Theorem 1.7 and our previous discussion in Subsection 1.3 we see that

$$\lim_{\gamma\downarrow 0}F_B(t\gamma^{-1};\gamma)=\begin{cases}\mathrm{e}^{-\frac{2}{\pi}t}, & t\geq 0\\ 0, & t<0\end{cases},\qquad \lim_{\gamma\downarrow 0}F_S(t\gamma^{-\frac{2}{3}};\gamma)=\begin{cases}1, & t>0\\ \mathrm{e}^{-\frac{2}{3\pi}|t|^{\frac{3}{2}}}, & t\leq 0\end{cases},\tag{1.39}$$

and

$$\lim_{\gamma\downarrow 0}\big(1-F_H(t\gamma^{-2},\alpha;\gamma)\big)=\begin{cases}1-\mathrm{e}^{-\frac{1}{\pi}\sqrt{t}}, & t\geq 0\\ 0, & t<0\end{cases}.\tag{1.40}$$

Hence by varying $\gamma\in[0,1]$ we interpolate between particle systems obeying random matrix theory statistics ($\gamma=1$) and systems that follow Poisson statistics (bulk) or Weibull statistics (soft-edge and hard-edge).

Remark 1.9. *The occurrence of (transformed) Weibull distribution functions in the limits of F_S and $1-F_H$ in (1.39) and (1.40) is consistent with the thinning process. We are effectively dealing with a sequence of independent random variables once $\gamma\downarrow 0$, and for such a sequence its extreme values (which are described by F_S and $1-F_H$) follow generically either Gumbel, Fréchet or Weibull statistics. For the same reason we shouldn't expect either of these three families in the limit $\gamma\downarrow 0$ for the bulk function F_B.*

Remark 1.10. *The usefulness of Proposition 1.6 in the derivation of tail expansions is showcased below for $\gamma\in[0,1)$ fixed, see Theorem 1.7 above. Once $\gamma\uparrow 1$ highly non-trivial and quasi-periodic transition regimes occur, see [18] for the transition between (1.33) and (1.36). It remains to be seen whether, say, (1.26) and Proposition 1.6 can simplify the derivation of the underlying transition asymptotics for F_B in [18].*

1.5. **Numerical comparison.** We offer a short comparison of the results given in Theorem 1.7 to the numerically computed values of $F_B(t;\gamma), F_S(t;\gamma)$ and $F_H(t,\alpha;\gamma)$. Those values were calculated by MATLAB implementing the algorithm given in [13], i.e. we discretize the relevant Fredholm determinants (1.22), (1.23) by the Nyström method using an m-point Gauss-Legendre quadrature rule. The results are shown in Figure 2 for the bulk gap function and in Figures 3, respectively 4, 5 and 6 for the extremal distribution functions, and we mention that $m=50$ quadrature points are sufficient to achieve a good matching between numerics and asymptotic results.

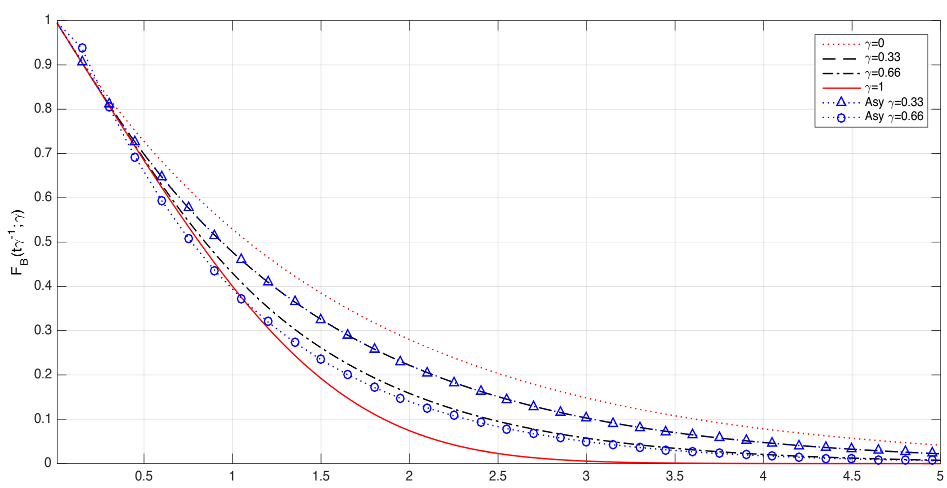

FIGURE 2. Plot of the gap function $F_B(t\gamma^{-1};\gamma)$ for various values of $\gamma\in[0,1]$. The result is computed with $m=50$ quadrature points and checked against (1.33) in blue with triangles for $\gamma=0.33$ and blue with circles for $\gamma=0.66$.

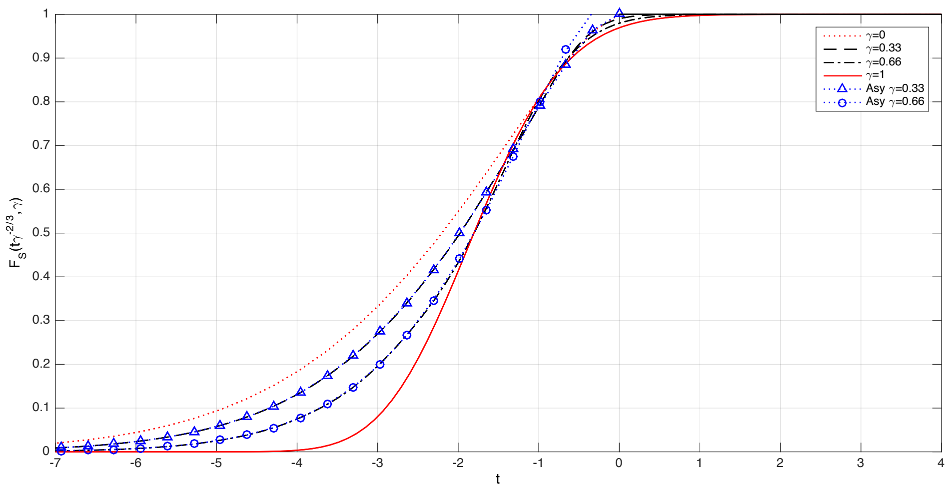

FIGURE 3. Plot of the distribution function $F_S(t\gamma^{-\frac{2}{3}};\gamma)$ for various values of $\gamma \in [0,1]$. The result is computed with $m = 50$ quadrature points and checked against (1.34) in blue with triangles for $\gamma = 0.33$ and blue with circles for $\gamma = 0.66$.

1.6. **The constant problem.** The exact evaluation of constant factors in asymptotic expansions of distribution, gap or correlation functions occurring in statistical mechanics or field theories is a long standing and challenging problem. The first rigorous solution of a constant problem for Painlevé equations (a special Painlevé III transcendent appearing in the Ising model) has been obtained in the work of Tracy [63]. Other constant problems have been studied in the works [4, 16, 52, 29, 30, 23, 22, 25, 54] and [3]. The tau functions that appear in all these papers correspond to very special families of Painlevé functions, and, as it has already been mentioned above, the success in their analysis was due to the presence of operator theoretical structures (Fredholm-, Toeplitz-, Hankel-determinants). The first results concerning the general two-parameter families of solutions of Painlevé equations have been obtained only recently in [42, 43]. These works are based on *conformal block representations* of isomonodromic tau functions — see [38, 39, 44]. Although very powerful, the conformal block approach still has to be put on rigorous ground. In the recent papers [48] and [47], it was shown that with the help of Riemann-Hilbert techniques the conjectural formulæ of [43] and [42] for the constant factor in the asymptotics of the Painlevé III and Painlevé VI tau functions can be proven. Moreover, a new result - the formula for the constant factor in the asymptotics of a generic Painlevé II tau function was established. In the work [55] the same technique has been used for solving the constant problem for the Painlevé I tau function. Finally, in the most recent work [37] in the case of the Painlevé VI, a determinant formula for the generic Painlevé VI tau function has been obtained, which also, in particular, provides a rigorous proof of the results of [42].

A central role in the constructions of papers [48] and [47] is played by an extension of the Jimbo-Miwa-Ueno differential to the full space of the extended monodromy data of the associated linear systems. This extension has been inspired by the works of Malgrange [56] and Bertola [7, 6], and, as a by-product, it has established a very interesting new fact about the Jimbo-Miwa-Ueno differential. It turns out that the original Jimbo-Miwa-Ueno differential form coincides up to a total derivative with the classical action differential. This in turn lead to Theorem 1.5 which, as we have already briefly explained, yields a new and much simpler way to derive the large gap asymptotics, as featured in this paper. In other words, a principal methodological message of our paper is that the original, very special, cases of tau functions have also benefited from the scheme that has been designed for the analysis of the general cases.

1.7. **Outline of paper.** A short outline for the remaining sections is as follows. We derive Theorem 1.3 and Theorem 1.5 in Section 2 through the use of Fredholm determinant formulæ for $F_B(t), F_S(t)$ and $F_H(t,\alpha)$ and straightforward differentiation. In Section 3 we outlined the above mentioned alternative proof of Theorem 1.5 based on the theory of isomonodromic tau functions. This alternative proof explains the origin and general theoretical meaning of the theorem's statement. Section 4 is devoted to the proof of (1.34) where we rely on the well-known Riemann-Hilbert representation [32] of the Ablowitz-Segur solution to Painlevé-II. The underlying Riemann-Hilbert problem is solved in [32] and we only require a minor extension of the known nonlinear steepest descent techniques ([32] focuses on $q(t;\gamma)$ only, but here we need $p(t;\gamma)$ and $\mathcal{H}_S(q,p,t)$ as well). Somewhat similar is our approach in the proof of (1.33) given in Section 5. The Riemann-Hilbert problem has been analyzed previously in [24, 17] and we simply read off $p(t;\gamma)$ and $\mathcal{H}_B(q,p,t)$. This changes in Section 6 when we address (1.35). The Riemann-Hilbert problem for $q(t,\alpha;\gamma)$ and $p(t,\alpha;\gamma)$ is known from [15] but was not asymptotically solved in the required scaling regime when $t\to+\infty$ and $\gamma\in[0,1)$ is fixed. For this reason we provide all necessary details following the roadmap of [26]: matrix factorizations, a g-function transformation, contour deformations, local model problems with Bessel and confluent hypergeometric functions and finally small norm estimates and iterations. The result of these steps is summarized in Theorem 6.13. After that we simply extract all relevant expansions and combine them in (1.28), leading to the final expansion (1.35).

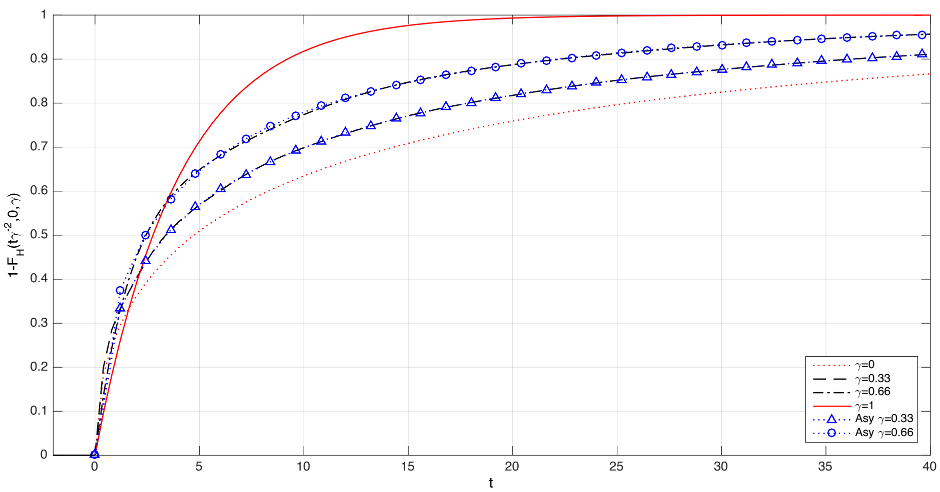

FIGURE 4. Plot of the distribution function $1-F_H(t\gamma^{-2},0;\gamma)$ for various values of $\gamma\in[0,1]$. The result is computed with $m=50$ quadrature points and checked against (1.35) for $\alpha=0$ in blue with triangles for $\gamma=0.33$ and blue with circles for $\gamma=0.66$.

2. Proof of Theorem 1.3 and Theorem 1.5

For the proof of Theorem 1.3 we shall rely on [34], Chapter 9.

Proof. Recall the well known [64, 65] Fredholm representations of the limiting distribution and gap functions in the complete Wishart ensemble,

$$F_B(t)=\det\big(1-K_{\sin}\restriction_{L^2(-t,t)}\big),\quad F_S(t)=\det\big(1-K_{\mathrm{Ai}}\restriction_{L^2(t,\infty)}\big),\quad F_H(t,\alpha)=\det\big(1-K_{\mathrm{Bess}}^{\alpha}\restriction_{L^2(0,t)}\big),$$

using the kernels from (1.24) and (1.25). Also, the limiting probability of having exactly $m\in\mathbb{Z}_{\geq 0}$ bulk, or soft-edge or hard-edge scaled eigenvalues $\mu_j^r, r=B,S,H$ in the interval $(-t,t)$, or (t,∞) or $(0,t)$ equals [34],

$$E_B(m,(-t,t))=\frac{(-1)^m}{m!}\frac{\partial^m}{\partial\xi^m}E_B((-t,t),\xi)\Big|_{\xi=1},\quad E_S(m,(t,\infty))=\frac{(-1)^m}{m!}\frac{\partial^m}{\partial\xi^m}E_S((t,\infty),\xi)\Big|_{\xi=1}$$

and

$$E_H(m,(0,t),\alpha)=\frac{(-1)^m}{m!}\frac{\partial^m}{\partial\xi^m}E_H((0,t),\xi,\alpha)\Big|_{\xi=1}$$

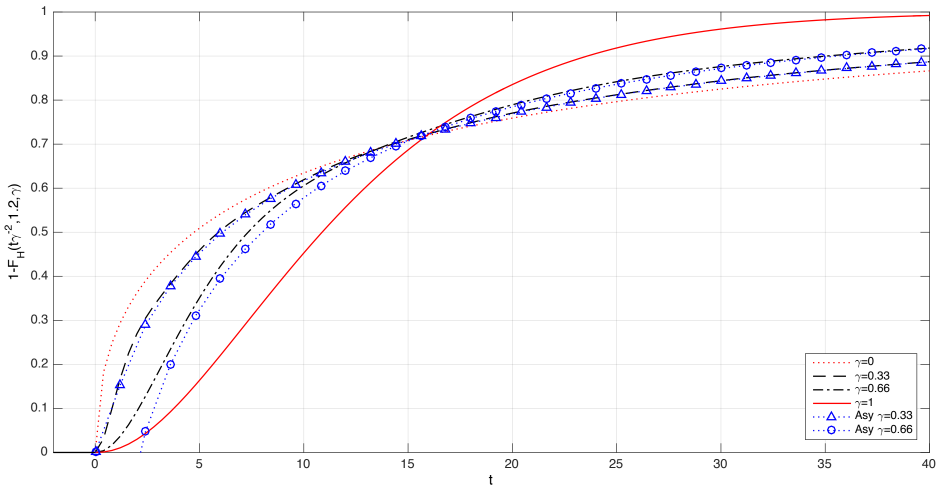

FIGURE 5. Plot of the distribution function $1 - F_H(t\gamma^{-2}, 1.2; \gamma)$ for various values of $\gamma \in [0, 1]$. The result is computed with $m = 50$ quadrature points and checked against (1.35) for $\alpha = 1.2$ in blue with triangles for $\gamma = 0.33$ and blue with circles for $\gamma = 0.66$.

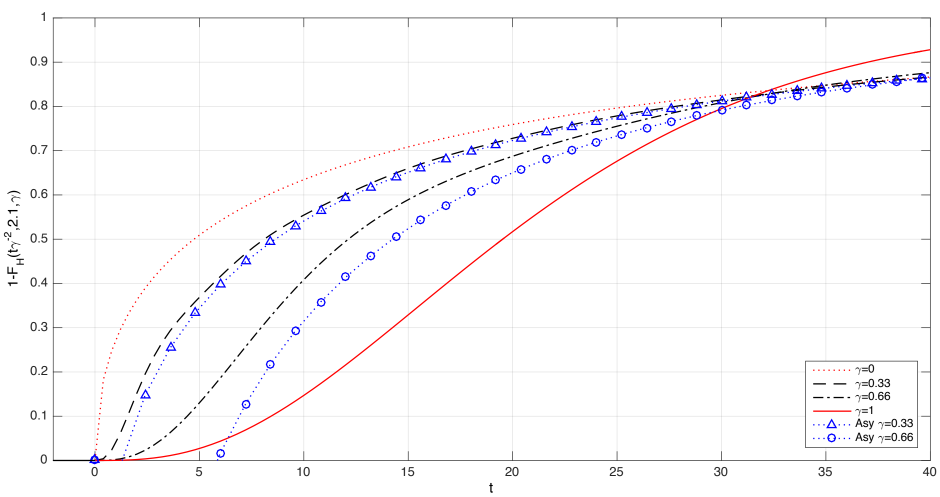

FIGURE 6. Plot of the distribution function $1 - F_H(t\gamma^{-2}, 2.1; \gamma)$ for various values of $\gamma \in [0, 1]$. The result is computed with $m = 50$ quadrature points and checked against (1.35) for $\alpha = 2.1$ in blue with triangles for $\gamma = 0.33$ and blue with circles for $\gamma = 0.66$.

in terms of the generating functions

$$E_B((-t,t),\xi) = \det\left(1 - \xi K_{\sin} \restriction_{L^2(-t,t)}\right), \qquad E_S((t,\infty),\xi) = \det\left(1 - \xi K_{\mathrm{Ai}} \restriction_{L^2(t,\infty)}\right) \tag{2.1}$$

and

$$E_H((0,t),\xi,\alpha) = \det\left(1 - \xi K^{\alpha}_{\mathrm{Bess}} \restriction_{L^2(t,\infty)}\right).$$

Returning to (1.16) (the case of the thinned extremal distributions is handled analogously) we have then

$$\begin{aligned} F_B(t,\gamma) = \sum_{m=0}^{\infty} E_B(m,(-t,t))(1-\gamma)^m &= \sum_{m=0}^{\infty} \frac{1}{m!}\frac{\partial^m}{\partial \xi^m} E_B((-t,t),\xi)\Big|_{\xi=1} (\gamma-1)^m \\ &= E_B((-t,t),\gamma) = \det\left(1 - \gamma K_{\sin} \restriction_{L^2(-t,t)}\right), \end{aligned}$$

where we used the definition of the incomplete Wishart ensemble in the first equality (each particle is removed independently with probability $1-\gamma$), identity (2.1) in the second and Taylor's theorem in the third. Since we have now established (1.22) and (1.23), the remainder of the proof (Hamiltonian representations and boundary conditions) follows at once from [34], Chapter 9: indeed, for the limiting gap function, use [34],(9.27) and Proposition 3.33. For the limiting distribution function of the largest eigenvalue after thinning, use [34], (9.26) and (9.43). Finally, for the limiting distribution function of the smallest eigenvalue after thinning, use [34], (9.62), (9.67) and (9.69). $\square$

We now address Theorem 1.5.

Proof. Take the t-derivative of the right hand side in (1.26),

$$\frac{\partial}{\partial t}\ln F_B(t;\gamma)=t\big(\mathcal{H}_B\big)_t-p_tq,$$

and now use the Hamiltonian system (1.7),

$$\frac{\partial}{\partial t}\ln F_B(t;\gamma)=t\big(\mathcal{H}_B\big)_t-p_tq=\mathcal{H}_B(q,p,t).$$

Thus, recall Proposition 1.3, left and right hand side in (1.26) can only differ by a t-independent constant. But from (1.17) we find $\mathcal{H}_{\mathrm{B}}(q,p,t)\sim-\frac{2\gamma}{\pi},t\downarrow 0$, and thus

$$\ln F_B(t;\gamma)=\int_0^t\mathcal{H}_B\big(q(s;\gamma),p(s;\gamma),s\big)\,\mathrm{d}s=-\frac{2\gamma t}{\pi}+\mathcal{O}\big(t^2\big),\quad t\downarrow 0,$$

which matches exactly the small t-behavior of the right hand side in (1.26), hence the aforementioned constant is zero. Next, take t-derivatives of the right hand side in (1.27),

$$\frac{\partial}{\partial t}\ln F_S(t;\gamma)=-\frac{1}{3}\mathcal{H}_S+\frac{2t}{3}\big(\mathcal{H}_S\big)_t-\frac{1}{3}p_tq+\frac{2}{3}pq_t,$$

and use the system (1.10) which leads to

$$\frac{\partial}{\partial t}\ln \mathrm{F}_S(t;\gamma)=\mathcal{H}_S(q,p,t).$$

But from (1.18),

$$\ln F_S(t;\gamma)=-\int_t^{\infty}\mathcal{H}_S\big(q(s;\gamma),p(s;\gamma),s\big)\,\mathrm{d}s=-\frac{\gamma}{16\pi}t^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}t^{\frac{3}{2}}}\big(1+o(1)\big),\quad t\to+\infty$$

which again matches the large positive t-behavior of the right hand side in (1.27), so the identity follows. Finally turn to (1.28) and take t-derivatives of both sides,

$$\frac{\partial}{\partial t}\ln F_H(t,\alpha;\gamma)=\mathcal{H}_H+2t\big(\mathcal{H}_H\big)_t-\frac{\alpha^2}{2t}\frac{q^2}{q^2-1}+pq_t,\qquad \alpha>-1.$$

But with (1.13) this implies that

$$\frac{\partial}{\partial t}\ln F_H(t,\alpha;\gamma)=\mathcal{H}_H,$$

and thus both sides in (1.28) can only differ by a t-independent term. As mentioned in Remark 1.1, the Hamiltonian is integrable on $(0,t)$ for $t>0$ and we find (compare Subsection 1.3),

$$\int_0^t\mathcal{H}_H\big(q(s,\alpha;\gamma),p(s,\alpha;\gamma),s,\alpha\big)\,\mathrm{d}s=-\frac{\gamma}{\Gamma^2(2+\alpha)}\left(\frac{t}{4}\right)^{\alpha+1}\big(1+o(1)\big),\quad t\downarrow 0. \tag{2.2}$$

On the other hand the integrands of $L(t,\alpha;\gamma)$ and $I_H(t,\alpha;\gamma)$, see (1.28), are singular at all points $t_k\in(0,+\infty)$ where $q^2(t_k,\alpha;\gamma)=1$. In fact the differential equation (A.23) leads to the Taylor expansion

$$q(t,\alpha;\gamma)=\pm1+d_k(t-t_k)+\mathcal{O}\big((t-t_k)^2\big),\qquad d_k^2t_k^2=\frac{1}{4}\alpha^2,\qquad |t-t_k|<r,$$

so that near t_k,

$$pq_t-\mathcal{H}_H(q,p,t,\alpha)=\pm\frac{\alpha^2}{4d_kt_k}\frac{1}{t-t_k}+\mathcal{O}(1),\qquad \frac{\alpha^2q^2}{2t(q^2-1)}=\pm\frac{\alpha^2}{4d_kt_k}\frac{1}{t-t_k}+\mathcal{O}(1),$$

and

$$\frac{\alpha^2}{2t(q^2-1)} = \pm\frac{\alpha^2}{4d_k t_k}\frac{1}{t-t_k} + \mathcal{O}(1).$$

For this reason we choose the path of integration for L and I_H in the right half-plane from $s=0$ to $s=t$ and we avoid all points t_k. With this choice, for $\alpha \geq 0$,

$$L(t,\alpha;\gamma) = -\frac{\alpha\gamma}{2}\left(\frac{t}{4}\right)^{\alpha}\frac{1}{\Gamma^2(1+\alpha)}\big(1+o(1)\big), \quad t\downarrow 0,$$

as well as

$$I_H(t,\alpha;\gamma) = -\frac{\alpha\gamma}{2}\left(\frac{t}{4}\right)^{\alpha}\big(1+o(1)\big) + \frac{\gamma}{\Gamma^2(2+\alpha)}\left(\frac{t}{4}\right)^{\alpha+1}\big(1+o(1)\big), \ \ t\downarrow 0.$$

This implies that

$$2t\,\mathcal{H}_H(q(t,\alpha;\gamma),p(t,\alpha;\gamma),t,\alpha) - L(t,\alpha;\gamma) + I_H(t,\alpha;\gamma) = \mathcal{O}\left(t^{\alpha+1}\right), \ \ t\downarrow 0; \quad \alpha\geq 0$$

which matches in turn the vanishing order in (2.2), i.e. the aforementioned t-independent term is identically zero for $\alpha \geq 0$. For $-1<\alpha<0$, we have instead, as $t\downarrow 0$,

$$L(t,\alpha;\gamma) = -\frac{\alpha}{2\gamma}\Gamma^2(1+\alpha)\left(\frac{t}{4}\right)^{-\alpha}\big(1+o(1)\big), \quad I_H(t,\alpha;\gamma) = \mathcal{O}\big(t^{-\alpha}\big) + \frac{\gamma}{\Gamma^2(2+\alpha)}\left(\frac{t}{4}\right)^{\alpha+1}\big(1+o(1)\big),$$

so again both sides in (1.28) vanish as $t\downarrow 0$, i.e. also for $-1<\alpha<0$ the identity holds true. □

3. Isomonodromic tau function and alternative proof of Theorem 1.5

As it has already been mentioned in our introduction, in this section we outline an alternative proof of Theorem 1.5 which is based on the Jimbo-Miwa-Ueno theory of the isomonodromic tau function. We will restrict ourselves to the soft edge case (1.26) only. The bulk and hard edge cases can be done in a similar way.

3.1. Lax system and classical Jimbo-Miwa-Ueno differential.

Consider the following 2×2 system of ordinary differential equations in the complex λ-plane,

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\lambda} = \left\{-4\mathrm{i}\lambda^2\sigma_3 + 4\mathrm{i}\lambda\begin{bmatrix}0 & q\\ -q & 0\end{bmatrix} + \begin{bmatrix}-\mathrm{i}t-2\mathrm{i}q^2 & -p\\ -p & \mathrm{i}t+2\mathrm{i}q^2\end{bmatrix}\right\}\mathbf{X} \equiv \mathbf{A}(\lambda,t)\mathbf{X}, \qquad \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{3.1}$$

where t,q,p are viewed as external parameters. This system has an irregular singular point at $\lambda=\infty$ of Poincaré rank 3 thus in turn (cf. [32]) seven canonical solutions $\{\mathbf{X}_j(\lambda),\lambda\in\Omega_j\}_{j=1}^7$ to (3.1) exist which are uniquely specified by the asymptotic expansion

$$\mathbf{X}_j(\lambda) \sim \widehat{\mathbf{X}}(\lambda)\mathrm{e}^{-\mathrm{i}(\frac{4}{3}\lambda^3+t\lambda)\sigma_3}, \ \ \lambda\to\infty, \ \lambda\in\Omega_j. \tag{3.2}$$

Here

$$\Omega_j = \left\{\lambda\in\mathbb{C}: \ \arg\lambda\in\left(\frac{\pi}{3}(j-2),\frac{\pi}{3}j\right)\right\}, \quad j=1,2,\ldots,7,$$

denote the canonical sectors, and $\widehat{\mathbf{X}}(\lambda)$ is the formal series,

$$\widehat{\mathbf{X}}(\lambda) = \mathbb{I} + \sum_{k=1}^{\infty}\frac{\mathbf{X}_k}{\lambda^k}, \tag{3.3}$$

whose matrix coefficients are explicitly expressed in terms of q and p; for instance,

$$\mathbf{X}_1 = \begin{bmatrix}-\frac{\mathrm{i}}{2}(\frac{p^2}{4}-tq^2-q^4) & \frac{q}{2}\\ \frac{q}{2} & \frac{\mathrm{i}}{2}(\frac{p^2}{4}-tq^2-q^4)\end{bmatrix}. \tag{3.4}$$

The space $\mathcal{M}$ of monodromy data of system (3.1) is generically two dimensional over the field of complex numbers, cf. [32], and it consists of the non-trivial entries in the *Stokes matrices*

$$\mathbf{S}_j = \mathbf{X}_j^{-1}(\lambda)\mathbf{X}_{j+1}(\lambda) = \begin{cases} \begin{bmatrix} 1 & 0 \\ s_j & 1 \end{bmatrix}, & j \equiv 1 \mod 2 \\ \begin{bmatrix} 1 & s_j \\ 0 & 1 \end{bmatrix}, & j \equiv 0 \mod 2 \end{cases},$$

which satisfy the following cyclic and symmetry constraints

$$\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3\mathbf{S}_4\mathbf{S}_5\mathbf{S}_6 = \mathbb{I}, \quad \sigma_2\mathbf{S}_j\sigma_2 = \mathbf{S}_{j+3}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

That is, the space $\mathcal{M}$ can be identified with the following affine cubic in $\mathbb{C}^3$,

$$\mathcal{M} = \left\{ s \equiv (s_1, s_2, s_3) \in \mathbb{C}^3 : s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0 \right\}.$$

The remarkable fact of the modern theory of Painlevé equations is that the Stokes parameters $s_j \equiv s_j(q,p,t)$ are the first integrals of the second Painlevé equation [31],

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = tq + 2q^3, \quad p = 2\frac{\mathrm{d}q}{\mathrm{d}t}. \tag{3.5}$$

Moreover, in terms of these integrals, the Ablowitz-Segur solution of (3.5) which we need in our study of the incomplete Wishart ensemble is characterized (see, e.g. [32]) by the equations,

$$s_1 = -s_3 = -s_4 = s_6 = -\mathrm{i}\sqrt{\gamma}, \quad s_2 = s_5 = 0.$$

This also means that, in the case of the Ablowitz-Segur family (1.18) for Painlevé-II, the space of monodromy data reduces to the complex plane $\mathbb{C}$,

$$\mathcal{M} = \{\gamma \in \mathbb{C}\}.$$

Another way to describe the relation of the linear system (3.1) to the Painlevé equation (3.5) is to say that the latter describes the *isomonodromic deformations* of the former. In fact, the dynamical system (3.5) is equivalent to the differential matrix equation,

$$\frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \mathbf{U}}{\partial \lambda} = [\mathbf{U}, \mathbf{A}]. \tag{3.6}$$

where

$$\mathbf{U}(\lambda, t) = -\mathrm{i}\lambda\sigma_3 + \mathrm{i}\begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}.$$

The nonlinear matrix equation (3.6) is usually called a zero curvature, or Lax equation, and it is a compatibility condition of two linear equations - system (3.1) and the t-differential equation

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{U}(\lambda, t)\mathbf{X}. \tag{3.7}$$

The pair of linear systems, (3.1) and (3.7) constitutes a *Lax pair* of the second Painlevé equation which was discovered by Flaschka and Newell in 1980 [31]. We are now passing to the isomonodromic tau function associated with this Lax pair.

The notion of isomonodromic tau functions was introduced by Jimbo, Miwa, and Ueno in 1980 in [50] for an arbitrary system of linear ordinary differential equations with rational coefficients. Their theory is based on a special 1-form defined on the space of the parameters of the system which is closed on the trajectories of the corresponding isomonodromy deformation equations. In the case of system (3.1), the Jimbo-Miwa-Ueno 1-form is defined by the equation (see [50], equation (5.1))

$$\omega_{\mathrm{JMU}} = -\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left\{ \left(\widehat{\mathbf{X}}(\lambda)\right)^{-1} \partial_\lambda \widehat{\mathbf{X}}(\lambda) \mathrm{d}_t \Theta(\lambda) \right\} \tag{3.8}$$

where

$$\Theta(\lambda) = -\mathrm{i}\left(\frac{4}{3}\lambda^3 + t\lambda\right)\sigma_3, \quad \text{and} \quad \mathrm{d}_t f := \frac{\partial f}{\partial t}\mathrm{d}t.$$

Using (3.3) and (3.4) one can easily transform (3.8) into

$$\omega_{\mathrm{JMU}} = \left(\frac{p^2}{4} - tq^2 - q^4\right) \mathrm{d}t \equiv \mathcal{H}_S(q,p,t)\,\mathrm{d}t. \tag{3.9}$$

Denote

$$\omega_{\mathrm{JMU}}(t;\gamma) \equiv \mathcal{H}_S(q(t;\gamma), p(t;\gamma), t)\,\mathrm{d}t$$

as the restriction of the form ω_{JMU} on the Ablowits-Segur solution of the Painevé II equation (3.5). The tau function corresponding to the Ablowits-Segur solution of the Painevé II equation (3.5) is then defined by the relation,

$$\mathrm{d}_t \ln \tau = \omega_{\mathrm{JMU}}(t;\gamma). \tag{3.10}$$

Comparing this with the equations stated in Theorem 1.3 , we see that the soft edge distribution function $F_S(t;\gamma)$ can be identified with the isomonodromic tau function corresponding to the Ablowitz-Segur Painlevé-II transcendent,

$$\tau(t;\gamma) \equiv F_S(t;\gamma). \tag{3.11}$$

In the next subsection we show how one can use (3.11) in the derivation of (1.27) in Theorem 1.5.

3.2. **Extended Jimbo-Miwa-Ueno differential.** In [50] it is also shown that the form ω_{JMU} can be alternatively defined as

$$\omega_{\mathrm{JMU}} = \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left\{\mathbf{A}(\lambda)\big(\mathrm{d}_t\widehat{\mathbf{X}}(\lambda)\big)\big(\widehat{\mathbf{X}}(\lambda)\big)^{-1}\right\}.$$

Following Section 4.2 of [47], where the generic two parameter family of the solutions of the second Painlevé equation is studied, we use this alternative definition and pass from the Jimbo-Miwa-Ueno form $\omega_{\mathrm{JMU}}(t;\gamma)$ to the following 1-form,

$$\omega_{\mathrm{ext}} = \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left\{\mathbf{A}(\lambda)\big(\mathrm{d}\widehat{\mathbf{X}}(\lambda)\big)\big(\widehat{\mathbf{X}}(\lambda)\big)^{-1}\right\}, \quad \mathrm{d} = \mathrm{d}_t + \mathrm{d}_\gamma, \tag{3.12}$$

defined on the extended space, $\{t\}\times\{\gamma\}$. Similar to the derivation of (3.9), we can substitute formula (3.3) for $\widehat{\mathbf{X}}(\lambda)$ into equation (3.12) and compute ω_{ext} explicitly in terms of p and q (cf. [47], (4.39)),

$$\omega_{\mathrm{ext}} = \mathcal{H}_S(q,p,t)\,\mathrm{d}t + \frac{2}{3}\left(pq_\gamma - \frac{1}{2}p_\gamma q - 2tq_\gamma\big(tq+2q^3\big) + \frac{t}{2}pp_\gamma\right)\mathrm{d}\gamma. \tag{3.13}$$

It should be noticed though that in order to arrive at this formula we now need, in addition to (3.4), the exact expressions for the matrix coefficients $\mathbf{X}_2$ and $\mathbf{X}_3$ which can be found in [47] - see equation (4.38). Two important facts about the form ω_{ext} can be extracted from (3.13):

- On the trajectories of the second Painlevé equation the form ω_{ext} coincides with the Jimbo-Miwa-Ueno form ω_{JMU}, i.e.,

$$\omega_{\mathrm{ext}}(t;\gamma=\mathrm{const.}) = \omega_{\mathrm{JMU}}(t;\gamma) \equiv d_t \ln F_S. \tag{3.14}$$

- The form ω_{ext} differs from the classical action differential, $p\mathrm{d}q - \mathcal{H}_S\,\mathrm{d}t$, by a total differential. Indeed, one can check by a direct differentiation that

$$\omega_{\mathrm{ext}} = \frac{1}{3}\mathrm{d}\big(2t\,\mathcal{H}_S - pq\big) + p\mathrm{d}q - \mathcal{H}_S\,\mathrm{d}t. \tag{3.15}$$

Restricting equation (3.15) to the Ablowitz-Segur trajectory $q = q(t;\gamma)$, $p = p(t;\gamma)$, $\gamma \equiv$ const. and taking into account equation (3.14) we arrive at the differential version of (1.27). The remaining two action formulæ (1.26) and (1.28) can be derived in a similar way using, instead of the Lax pair (3.1), (3.7), the Lax pairs corresponding to the dynamical systems (1.7) and (1.13), respectively.

We complete our presentation of this alternative proof of Theorem 1.5 by showing that the transformation of equation (3.13) into equation (3.15) is not an accident. In fact, there is a deep reason why this transformation takes place. To this end, let us consider the form ω_{ext} on the whole extended monodromy space, $\{t\}\times\mathcal{M}$, i.e. we pass from the one parameter Ablowitz-Segur family of solutions to Painlevé-II to the general two parameter set of solutions,

$$q \equiv q(t;s_1,s_2), \quad p \equiv p(t;s_1,s_2),$$

(we chose s_1 and s_2 as the local coordinates on $\mathcal{M}$). This means, that the differentiation d in (3.12) now means $\mathrm{d} = \mathrm{d}_t + \mathrm{d}_{s_1} + \mathrm{d}_{s_2}$ and equation (3.13) is replaced by the whole equation (4.39) of [47],

$$\omega_{\mathrm{ext}} = \mathcal{H}_S(q,p,t)\,\mathrm{d}t + \frac{2}{3}\left(pq_{s_1} - \frac{1}{2}p_{s_1}q - 2tq_{s_1}\big(tq + 2q^3\big) + \frac{t}{2}pp_{s_1}\right)\mathrm{d}s_1$$
$$+ \frac{2}{3}\left(pq_{s_2} - \frac{1}{2}p_{s_2}q - 2tq_{s_2}\big(tq + 2q^3\big) + \frac{t}{2}pp_{s_2}\right)\mathrm{d}s_2. \tag{3.16}$$

The general key fact about the extended form ω_{ext} is that its external derivative is a 2-form on $\mathcal{M}$ and it does not depend on t. In fact, one can check directly that (cf. (4.48) in [47])

$$\mathrm{d}\omega_{\mathrm{ext}} = \big(p_{s_1}q_{s_2} - p_{s_2}q_{s_1}\big)\mathrm{d}s_1 \wedge \mathrm{d}s_2 \equiv \Omega,$$

where Ω is the canonical symplectic form on the phase space $\{(p,q)\}$. A classical fact of Hamiltonian mechanics is that the external derivative of the classical action differential equals the same symplectic form,

$$\mathrm{d}\big(p\mathrm{d}q - \mathcal{H}_S\,\mathrm{d}t\big) = \big(p_{s_1}q_{s_2} - p_{s_2}q_{s_1}\big)\mathrm{d}s_1 \wedge \mathrm{d}s_2 \equiv \Omega$$

Therefore,

$$\omega_{\mathrm{ext}} - \big(p\mathrm{d}q - \mathcal{H}_S\big)\,\mathrm{d}t = \text{total differential}$$

The fact that this total differential equals $\frac{1}{3}\mathrm{d}\big(2t\,\mathcal{H}_S - pq\big)$ is the result of a concrete calculation. We do not yet have a conceptual way to find this differential.

4. Proof of Theorem 1.7, expansion (1.34)

It is well known, cf. [32], that we can characterize the functions (q,p) in (1.10), (1.18) through the solution of the following Riemann-Hilbert problem (RHP)

Riemann-Hilbert Problem 4.1. *Let $t \in \mathbb{R}$ and $\gamma \in [0,1]$. Determine the piecewise analytic function $\mathbf{Y} = \mathbf{Y}(\lambda;t,\gamma) \in \mathbb{C}^{2\times 2}$ such that*

(1) *$Y(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash(\Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_6)$ with the four rays*

$$\Gamma_1 = \left\{\lambda \in \mathbb{C}:\quad \arg\lambda = \frac{\pi}{6}\right\},\qquad \Gamma_3 = \left\{\lambda \in \mathbb{C}:\quad \arg\lambda = \frac{5\pi}{6}\right\}$$
$$\Gamma_4 = \left\{\lambda \in \mathbb{C}:\quad \arg\lambda = \frac{7\pi}{6}\right\},\qquad \Gamma_6 = \left\{\lambda \in \mathbb{C}:\quad \arg\lambda = \frac{11\pi}{6}\right\}$$

oriented from the origin $\lambda = 0$ towards infinity, compare Figure 7 below.

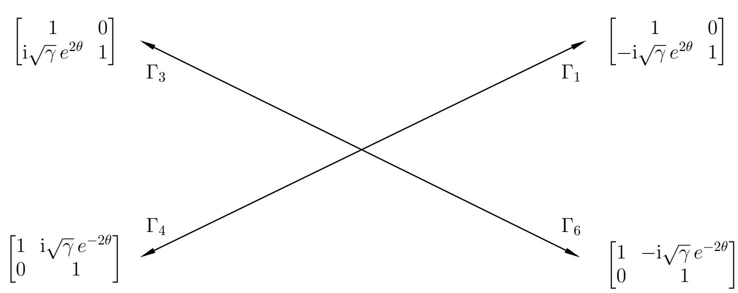

Figure 7. The oriented jump contours for the master function $\mathbf{Y}(\lambda;t,\gamma)$ of RHP 4.1 in the complex λ-plane.

(2) *The boundary values $\mathbf{Y}_+(\lambda)$ (or $\mathbf{Y}_-(\lambda)$) from the left (or right) side of the oriented contour Γ_k satisfy the jump relation*

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\mathrm{e}^{-\theta(\lambda,t)\sigma_3}\mathbf{S}_k\mathrm{e}^{\theta(\lambda,t)\sigma_3},\quad \lambda \in \Gamma_k,\quad k = 1,2,3,4$$

with

$$\theta(\lambda,t) = \mathrm{i}\left(\frac{4}{3}\lambda^3 + t\lambda\right),\qquad \sigma_3 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right];$$

and the λ-independent matrices

$$\mathbf{S}_1=\left[\begin{smallmatrix}1&0\\-\mathrm{i}\sqrt{\gamma}&1\end{smallmatrix}\right],\quad \mathbf{S}_3=\left[\begin{smallmatrix}1&0\\\mathrm{i}\sqrt{\gamma}&1\end{smallmatrix}\right],\quad \mathbf{S}_4=\left[\begin{smallmatrix}1&\mathrm{i}\sqrt{\gamma}\\0&1\end{smallmatrix}\right]\quad \mathbf{S}_6=\left[\begin{smallmatrix}1&-\mathrm{i}\sqrt{\gamma}\\0&1\end{smallmatrix}\right].$$

(3) *As $\lambda\to\infty$, $\mathbf{Y}(\lambda)$ is normalized in the following way*

$$\mathbf{Y}(\lambda)=\mathbb{I}+\mathbf{Y}_1\lambda^{-1}+\mathbf{Y}_2\lambda^{-2}+\mathcal{O}\left(\lambda^{-3}\right),\qquad \mathbf{Y}_\ell=\big(Y_\ell^{jk}\big)_{j,k=1}^2$$

As proven in [12], the latter problem for $\mathbf{Y}(\lambda)$ is uniquely solvable for all $t\in\mathbb{R},\gamma\in[0,1]$ and its solution determines the Ablowitz-Segur transcendents via

$$q(t;\gamma)=2Y_1^{12},\quad p(t;\gamma)=2q_t(t;\gamma)=-8\mathrm{i}\big(Y_2^{12}+Y_1^{12}Y_1^{11}\big); \tag{4.1}$$

Moreover the Hamiltonian function $\mathcal{H}_S=\mathcal{H}_S(q,p,t)$ can be read off directly from RHP 4.1 as well,

$$\mathcal{H}_S\big(q(t;\gamma),p(t;\gamma),t\big)=2\mathrm{i}Y_1^{11}. \tag{4.2}$$

The Riemann-Hilbert representation (4.1) has been used numerous times in the literature to derive the leading asymptotic behavior of $q(t;\gamma)$ as $t\to\pm\infty$ and $\gamma\in[0,1]$ is kept fixed, cf. [32] for more on the history of this subject. For our purposes (i.e. the proof of Theorem 1.7, expansion (1.34)) the estimates given in [32] have to be slightly extended. With this goal in mind we shall not reproduce all steps carried out in [32], instead we only provide references and jump immediately to the key estimates.

4.1. **Nonlinear steepest descent analysis for RHP 4.1 (in a nutshell).** Our goal is to solve RHP 4.1 for $\mathbf{Y}(\lambda;t,\gamma)\in\mathbb{C}^{2\times2}$ for all values $(-t,v)\in\mathbb{R}_+^2$ such that

$$(-t)\geq t_0,\quad\text{and}\quad 0\leq v=-\ln(1-\gamma)<+\infty\ \text{ is fixed.} \tag{4.3}$$

This is achieved by first rescaling the initial function with the large parameter, $\mathbf{X}(\lambda)=\mathbf{Y}(\lambda\sqrt{-t}\,),\lambda\in\mathbb{C}\backslash(\Gamma_1\cup\Gamma_3\cup\Gamma_4\cup\Gamma_6)$. Secondly, contour deformations $\mathbf{X}(\lambda)\mapsto\mathbf{T}(\lambda)$, see [32], Figure 9.4 and thirdly, matrix factorizations and opening of lens transformations $\mathbf{T}(\lambda)\mapsto\mathbf{S}(\lambda)$, see [32], (9.4.7) and Figures 9.5, 9.6. After those initial three transformations the RHP for $\mathbf{S}(\lambda)$ is already in a localized state since the underlying jump matrix $\mathbf{G}_\mathbf{S}(\lambda;t,v)$ obeys (see [32], (9.4.30))

$$\mathbf{G}_\mathbf{S}(\lambda;t,v)=\mathbb{I}+\mathcal{O}\left(\mathrm{e}^{v-4(-t)^{\frac{3}{2}}|\lambda\mp\frac{1}{2}|^2}\right),\qquad(-t)\geq t_0, \tag{4.4}$$

for λ along the eight contours shown in [32], Figure 9.6 that extend to infinity. Hence one needs to focus only on the line segment $(-\frac{1}{2},\frac{1}{2})\subset\mathbb{R}$ and two small vicinities of the endpoints $\pm\frac{1}{2}$. But all parametrices are well known, e.g. for the segment (see [32], (9.4.8)) we take

$$\mathbf{P}^{(\infty)}(\lambda)=\left(\frac{\lambda+\frac{1}{2}}{\lambda-\frac{1}{2}}\right)^{\nu\sigma_3},\qquad \lambda\in\mathbb{C}\Big\backslash\left[-\frac{1}{2},\frac{1}{2}\right];\qquad \nu=\frac{v}{2\pi\mathrm{i}}\in\mathrm{i}\mathbb{R},$$

and for the neighborhoods of $\lambda=\pm\frac{1}{2}$ standard parabolic cylinder functions [60] come into play. We shall denote those parametrices by $\mathbf{P}^{(\frac{1}{2})}(\lambda)$, see [32], (9.4.20) and $\mathbf{P}^{(-\frac{1}{2})}(\lambda)$, compare [32], (9.4.24). The three explicit model functions $\mathbf{P}^{(\infty)}(\lambda),\mathbf{P}^{(\frac{1}{2})}(\lambda)$ and $\mathbf{P}^{(-\frac{1}{2})}(\lambda)$ are then compared locally to the unknown $\mathbf{S}(\lambda)$,

$$\mathbf{R}(\lambda)=\mathbf{S}(\lambda)\begin{cases}\big(\mathbf{P}^{(\frac{1}{2})}(\lambda)\big)^{-1}, & \lambda\in\mathbb{D}_r(\frac{1}{2})\\ \big(\mathbf{P}^{(-\frac{1}{2})}(\lambda)\big)^{-1}, & \lambda\in\mathbb{D}_r(-\frac{1}{2})\\ \big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, & \lambda\notin\big(\mathbb{D}_r(\frac{1}{2})\cup\mathbb{D}_r(-\frac{1}{2})\big)\end{cases},\qquad \mathbb{D}_r(\lambda_0)=\{\lambda\in\mathbb{C}:\ |\lambda-\lambda_0|<r\}, \tag{4.5}$$

with fixed radius $0<r<\frac{1}{8}$. Recalling the model function properties we obtain the following RHP for the ratio function $\mathbf{R}(\lambda)$.

Riemann-Hilbert Problem 4.2. *Find $\mathbf{R}(\lambda)=\mathbf{R}(\lambda;t,v)\in\mathbb{C}^{2\times2}$ with $(-t,v)\in\mathbb{R}_+^2$ such that*

(1) *$\mathbf{R}(\lambda)$ is analytic for $\lambda\in\mathbb{C}\backslash\Sigma_\mathbf{R}$ where $\Sigma_\mathbf{R}=\partial\mathbb{D}_r(-\frac{1}{2})\cup\partial\mathbb{D}_r(\frac{1}{2})\cup\Sigma_\infty$ is shown in Figure 8 below.*

(2) *Along the contour $\Sigma_\mathbf{R}$ we have jumps $\mathbf{R}_+(\lambda)=\mathbf{R}_-(\lambda)\mathbf{G}_\mathbf{R}(\lambda;t,\gamma),\lambda\in\Sigma_\mathbf{R}$ with*

$$\mathbf{G}_\mathbf{R}(\lambda;t,v)=\mathbf{P}^{(\infty)}(\lambda)\mathbf{G}_\mathbf{S}(\lambda;t,v)\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1},\quad \lambda\in\Sigma_\infty$$

and

$$\mathbf{G}_\mathbf{R}(\lambda;t,v)=\mathbf{P}^{(\pm\frac{1}{2})}(\lambda)\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1},\quad \lambda\in\partial\mathbb{D}_r\left(\pm\frac{1}{2}\right).$$

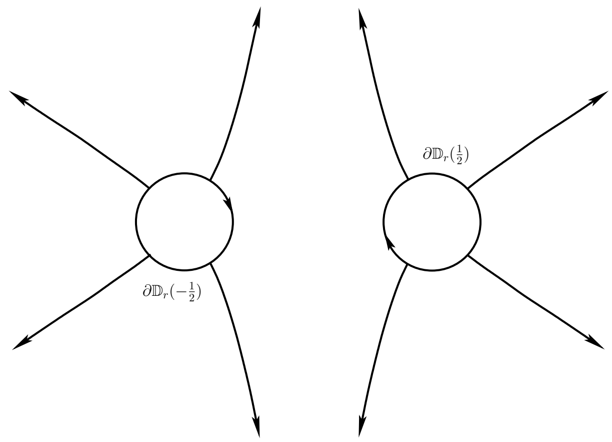

FIGURE 8. The oriented jump contours for the ratio function $\mathbf{R}(\lambda)$ in the complex λ-plane. The eight contours extending to infinity are summarized as Σ_∞.

By construction, there are no jumps in the interior of $\mathbb{D}_r(\pm\frac{1}{2})$ *and along* $[-\frac{1}{2},\frac{1}{2}]$.

(3) *As* $\lambda\to\infty$,
$$\mathbf{R}(\lambda)=\mathbb{I}+\mathcal{O}\left(\lambda^{-1}\right).$$

We now see how the constraint (4.3) guarantees that all jump matrices in RHP 4.2 are close to the identity in the same scaling regime. First turn towards Σ_∞: from (4.4) and the fact that $\nu\in\mathrm{i}\mathbb{R}$ we obtain at once,

Proposition 4.3. *There exist constants* $t_0>0$ *and* $c>0$ *such that*
$$\|\mathbf{G}_{\mathbf{R}}(\cdot;t,v)-\mathbb{I}\|_{L^2\cap L^\infty(\Sigma_\infty)}\leq c\,\mathrm{e}^{2v-4(-t)^{\frac{3}{2}}r^2},\qquad\forall\,(-t)\geq t_0,\quad v\geq 0.$$
The parameter $0<r<\frac{1}{8}$ *has been introduced previously in* (4.5).

Second, for $\partial\mathbb{D}_r(\pm\frac{1}{2})$ we recall [32], (9.4.23) and (9.4.33),

Proposition 4.4. *For any fixed* $v\in[0,+\infty)$ *there exist positive constants* $t_0=t_0(v)$ *and* $c=c(v)$ *such that*
$$\|\mathbf{G}_{\mathbf{R}}(\cdot;t,v)-\mathbb{I}\|_{L^2\cap L^\infty(\partial\mathbb{D}_r(\pm\frac{1}{2}))}\leq c\,(-t)^{-\frac{3}{4}},\qquad\forall\,(-t)\geq t_0.$$

By general theory, cf. [26], the last two estimates ensures unique solvability of the ratio problem 4.2 in the scaling regime (4.3), in fact

Theorem 4.5. *For any fixed* $v\in[0,+\infty)$ *there exist* $t_0=t_0(v)>0$ *and* $c=c(v)>0$ *such that the ratio problem 4.2 is uniquely solvable in* $L^2(\Sigma_{\mathbf{R}})$ *for all* $(-t)\geq t_0$. *We can compute its solution iteratively from the integral equation*
$$\mathbf{R}(\lambda)=\mathbb{I}+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_{\mathbf{R}}}\mathbf{R}_-(w)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\frac{\mathrm{d}w}{w-\lambda},\quad\lambda\in\mathbb{C}\backslash\Sigma_{\mathbf{R}},$$
using that
$$\|\mathbf{R}_-(\cdot;s,v)-\mathbb{I}\|_{L^2(\Sigma_{\mathbf{R}})}\leq c\,(-t)^{-\frac{3}{4}},\ \ \forall\,(-t)\geq t_0.$$

At this point we can extract all relevant asymptotic information via (4.1) and (4.2).

4.2. **Extraction of asymptotics and proof of expansion** (1.34)**.** Tracing back all explicit and invertible transformations, i.e. the sequence
$$\mathbf{Y}(\lambda)\mapsto\mathbf{X}(\lambda)\mapsto\mathbf{T}(\lambda)\mapsto\mathbf{S}(\lambda)\mapsto\mathbf{R}(\lambda),$$
we obtain the following formulæ,
$$\mathbf{Y}_1=\sqrt{-t}\left(\nu\sigma_3+\frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\mathbf{R}_-(w)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\mathrm{d}w\right),$$

and

$$\mathbf{Y}_2 = -t\left(\frac{\nu^2}{2}\mathbb{I} + \frac{\mathrm{i}\nu}{2\pi}\int_{\Sigma_{\mathbf{R}}}\mathbf{R}_-(w)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\mathrm{d}w\,\sigma_3 + \frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\mathbf{R}_-(w)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)w\,\mathrm{d}w\right).$$

We begin with the asymptotic estimation of the integrals

$$\mathbf{J} = \frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\,\mathrm{d}w,\quad \mathbf{J} = \big(J^{jk}\big)_{j,k=1}^2; \qquad \mathbf{K} = \frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)w\,\mathrm{d}w, \qquad \mathbf{K} = \big(K^{jk}\big)_{j,k=1}^2$$

using [32], (9.4.31), (9.4.32) and the residue theorem.

Lemma 4.6. *As $t\to-\infty$, with $s=(-t)^{\frac{3}{2}}$,*

$$J^{11} = -\frac{\mathrm{i}\nu^2}{12s} + \mathcal{O}\left(s^{-2}\right), \quad J^{12} = \frac{1}{2}\sqrt{\frac{v}{\pi s}}\cos\big(\phi(s,v)\big) + \mathcal{O}\big(s^{-\frac{3}{2}}\big),$$

and

$$K^{12} = \frac{\mathrm{i}}{4}\sqrt{\frac{v}{\pi s}}\sin\big(\phi(s,v)\big) + \mathcal{O}\big(s^{-\frac{3}{2}}\big); \qquad \phi(s,v) = \frac{2}{3}s - \frac{v}{2\pi}\ln(8s) + \frac{\pi}{4} - \arg\Gamma\left(\frac{v}{2\pi\mathrm{i}}\right).$$

All error terms are uniform with respect to v chosen from compact subsets of $[0,+\infty)$.

Next we obtain from Theorem 4.5 that for $w\in\Sigma_R$,

$$\mathbf{R}_-(w)-\mathbb{I} = \frac{1}{2\pi\mathrm{i}}\int_{\Sigma_{\mathbf{R}}}\big(\mathbf{G}_{\mathbf{R}}(\mu)-\mathbb{I}\big)\frac{\mathrm{d}\mu}{\mu-w_-} + \mathcal{O}\big((-t)^{-\frac{3}{2}}\big),$$

and thus, iterating once, where

$$\mathbf{L} = \frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\big(\mathbf{R}_-(w)-\mathbb{I}\big)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\,\mathrm{d}w, \qquad \mathbf{L} = \big(L^{jk}\big)_{j,k=1}^2,$$

Lemma 4.7. *As $t\to-\infty$ with $s=(-t)^{\frac{3}{2}}$,*

$$L^{11} = -\frac{2\mathrm{i}\nu^2}{3s} + \frac{\nu}{4s}\sin\big(2\phi(s,v)\big) + \mathcal{O}\big(s^{-\frac{3}{2}}\big),$$

and the error term is uniform with respect to v chosen from compact subsets of $[0,+\infty)$.

Now we go back to (4.1) and (4.2),

Corollary 4.8. *As $t\to-\infty$, with $s=(-t)^{\frac{3}{2}}$ and fixed $v\in[0,+\infty)$,*

$$q(t;\gamma) = (-t)^{-\frac{1}{4}}\sqrt{\frac{v}{\pi}}\cos\big(\phi(s,v)\big) + \mathcal{O}\left((-t)^{-\frac{7}{4}}\right), \quad p(t;\gamma) = 2(-t)^{\frac{1}{4}}\sqrt{\frac{v}{\pi}}\sin\big(\phi(s,v)\big) + \mathcal{O}\big((-t)^{-\frac{1}{2}}\big), \tag{4.6}$$

and

$$\mathcal{H}_S\big(q(t;\gamma),p(t;\gamma),t\big) = \frac{v}{\pi}\sqrt{-t} + \frac{3v^2}{8\pi^2 t} - \frac{v}{4\pi t}\sin\big(2\phi(s,v)\big) + \mathcal{O}\big((-t)^{-\frac{7}{4}}\big).$$

The last result allows us to determine all t-dependent leading terms, compare Proposition 1.3

Corollary 4.9. *As $t\to-\infty$,*

$$\ln F_S(t;\gamma) = -\frac{2v}{3\pi}|t|^{\frac{3}{2}} + \frac{v^2}{4\pi^2}\ln\big(|t|^{\frac{3}{2}}\big) + D(v) + \mathcal{O}\big(|t|^{-\frac{3}{4}}\big),$$

where $D(v)$ is t-independent and the error is uniform with respect to v chosen from compact subset of $[0,+\infty)$.

As for $D(v)$, we now use (1.27) and Corollary 1.6. First, as $t\to-\infty$,

$$\frac{1}{3}(2t\,\mathcal{H}_S - pq) = -\frac{2v}{3\pi}|t|^{\frac{3}{2}} + \frac{v^2}{4\pi^2} - \frac{v}{2\pi}\sin\big(2\phi(s,v)\big) + \mathcal{O}\big(|t|^{-\frac{3}{4}}\big).$$

And second, with (4.6),

$$\frac{\partial I_S}{\partial\gamma} = pq_\gamma = \frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\frac{v}{2\pi}\sin\big(2\phi(s,v)\big) + \frac{v^2}{4\pi^2}\ln(8s)\right) - \frac{v}{\pi}\frac{\mathrm{d}}{\mathrm{d}\gamma}\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right) + \mathcal{O}\left(|t|^{-\frac{3}{2}}\ln|t|\right)$$

so that all together (since $F_S(t;0)=1$),

Proposition 4.10. *As $t \to -\infty$,*

$$\ln F_S(t;\gamma) = -\frac{2v}{3\pi}|t|^{\frac{3}{2}} + \frac{v^2}{4\pi^2}\ln\left(8|t|^{\frac{3}{2}}\right) + \frac{v^2}{4\pi^2} - \frac{1}{\pi}\int_0^{\gamma} v(\gamma')\frac{\mathrm{d}}{\mathrm{d}\gamma'}\arg\Gamma\left(\frac{\mathrm{i}v(\gamma')}{2\pi}\right)\mathrm{d}\gamma' + \mathcal{O}\big(|t|^{-\frac{3}{4}}\big)$$

uniformly for $\gamma \in [0,1)$ chosen from compact subsets.

We now only have to recall the following standard property of the Barnes G-function, cf. [60] ,

$$\int_0^z \ln\Gamma(1+t)\,\mathrm{d}t = \frac{z}{2}\ln(2\pi) - \frac{z}{2}(z+1) + z\ln\Gamma(1+z) - \ln G(1+z), \qquad z \in \mathbb{C}: \quad \Re z > -1 \tag{4.7}$$

and Theorem 1.7, expansion (1.34) follows at once.

5. Proof of Theorem 1.7, expansion (1.33)

It is known from [24], Section 4 that we can characterize the functions (q,p) in (1.7), (1.17) through the solution of the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 5.1. *Let $t \in \mathbb{R}_{\geq 0}$ and $\gamma \in [0,1]$. Determine the piecewise analytic function $\mathbf{Y} = \mathbf{Y}(\lambda;t,\gamma) \in \mathbb{C}^{2\times 2}$ such that*

(1) *$\mathbf{Y}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[-1,1]$ with the line segment $[-1,1]$ oriented from left to right as shown in Figure 9 below.*

(2) *The boundary values $\mathbf{Y}_+(\lambda)$ (or $\mathbf{Y}_-(\lambda)$) from the left (or right) side of the oriented contour $(-1,1)$ obey the jump relation*

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\mathrm{e}^{\mathrm{i}t\lambda\sigma_3}\begin{bmatrix} 1-\gamma & \gamma \\ -\gamma & 1+\gamma \end{bmatrix}\mathrm{e}^{-\mathrm{i}t\lambda\sigma_3}, \quad \lambda \in (-1,1). \tag{5.1}$$

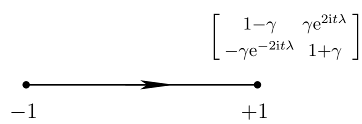

Figure 9. The oriented jump contours for the master function $\mathbf{Y}(\lambda;t,\gamma)$ of RHP 5.1 in the complex λ-plane.

(3) *Near the endpoints $\lambda = \pm 1$, we have the singular behavior*

$$\mathbf{Y}(\lambda) = \widehat{\mathbf{Y}}(\lambda)\left\{I + \frac{\gamma}{2\pi\mathrm{i}}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\ln\left(\frac{\lambda-1}{\lambda+1}\right)\right\}\mathrm{e}^{-\mathrm{i}t\lambda\sigma_3}, \quad \lambda \to \pm 1$$

where $\ln$ denotes the principal branch of the logarithm and $\widehat{\mathbf{Y}}(\lambda)$ is analytic at $\lambda = \pm 1$.

(4) *As $\lambda \to \infty$, $\mathbf{Y}(\lambda)$ is normalized as*

$$\mathbf{Y}(\lambda) = \mathbb{I} + \mathbf{Y}_1\lambda^{-1} + \mathcal{O}\left(\lambda^{-2}\right), \quad \mathbf{Y}_\ell = \left(Y_\ell^{jk}\right)_{j,k=1}^2.$$

As proven in [45, 49], the latter problem for $\mathbf{Y}(\lambda)$ is uniquely solvable for all $t \geq 0, \gamma \in [0,1]$ and its solution determines the Jimbo-Miwa-Mori-Sato transcendents via

$$q(t;\gamma) = \frac{1}{2}Y_1^{11} + \mathrm{i}t\big(Y_1^{12}\big)^2, \qquad \sinh\left(\frac{1}{2}p(t;\gamma)\right) = -\frac{\mathrm{i}tY_1^{12}}{q(t;\gamma)}. \tag{5.2}$$

We have in addition for the Hamiltonian function

$$\mathcal{H}_B\big(q(t;\gamma), p(t;\gamma), t\big) = -2\mathrm{i}Y_1^{11}, \tag{5.3}$$

and equations (5.2), (5.3) are the starting point for our asymptotic analysis. The Riemann-Hilbert problem 5.1 was solved asymptotically for $t \to +\infty$ and $\gamma \in [0,1)$ (fixed and for certain moving values of $\gamma \uparrow 1$) in [17][1]. Similar to Section 4 this allows us to save time and space.

[1]The reference [17] uses $v = -\frac{1}{2}\ln(1-\gamma)$ instead of $v = -\ln(1-\gamma)$. This has to be remembered in Subsection 5.1.

5.1. **Nonlinear steepest descent analysis for RHP 5.1 (in a nutshell).** Our goal is to solve RHP 5.1 for $\mathbf{Y}(\lambda;t,\gamma)\in\mathbb{C}^{2\times 2}$ for all values $(t,v)\in\mathbb{R}_+^2$ such that

$$t\geq t_0,\qquad \text{and}\quad 0\leq v=-\ln(1-\gamma)<+\infty\quad\text{is fixed.}\tag{5.4}$$

To achieve this we first use matrix factorizations and an opening of lens transformation, $\mathbf{Y}(\lambda)\mapsto\mathbf{S}(\lambda)$, see [17], Figure 1 and RHP 2.2. After this step the problem is already localized since the jump matrix $\mathbf{G}_{\mathbf{S}}(\lambda;t,v)$ obeys (see [17], page 218 top)

$$\mathbf{G}_{\mathbf{S}}(\lambda;t,v)=\mathbb{I}+\mathcal{O}\left(\mathrm{e}^{v-2t|\Im\lambda|}\right),\qquad t\geq t_0\tag{5.5}$$

for λ on the contours in the upper and lower half-plane, see [17], Figure 1. Hence we address the local problems on $(-1,1)\subset\mathbb{R}$ and in the vicinities of the endpoints ±1. The parametrices are again standard,

$$\mathbf{P}^{(\infty)}(\lambda)=\left(\frac{\lambda+1}{\lambda-1}\right)^{\nu\sigma_3},\quad\lambda\in\mathbb{C}\backslash[-1,1];\qquad \nu=\frac{v}{2\pi\mathrm{i}}\in\mathrm{i}\mathbb{R}$$

is chosen for the segment (see [17], (2.1)) and near $\lambda=\pm1$ confluent hypergeometric functions come into play. Let $\mathbf{P}^{(1)}(\lambda)$ and $\mathbf{P}^{(-1)}(\lambda)$ denote the required model functions, see [17], (2.4) and (2.6). These functions are unimodular and can be compared to the above $\mathbf{S}(\lambda)$,

$$\mathbf{R}(\lambda)=\mathbf{S}(\lambda)\begin{cases}\big(\mathbf{P}^{(1)}(\lambda)\big)^{-1}, & \lambda\in\mathbb{D}_r(1)\\ \big(\mathbf{P}^{(-1)}(\lambda)\big)^{-1}, & \lambda\in\mathbb{D}_r(-1)\\ \big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, & \lambda\notin\big(\mathbb{D}_r(1)\cup\mathbb{D}_r(-1)\big)\end{cases},$$

where $0<r<\frac{1}{4}$ is kept fixed, see [17], (2.8). In turn we find the problem outlined below.

Riemann-Hilbert Problem 5.2. *The function $\mathbf{R}(\lambda)=\mathbf{R}(\lambda;t,v)\in\mathbb{C}^{2\times2}$ has the following properties.*

(1) $\mathbf{R}(\lambda)$ *is analytic for $\lambda\in\mathbb{C}\backslash\Sigma_{\mathbf{R}}$ where $\Sigma_{\mathbf{R}}=\partial\mathbb{D}_r(-1)\cup\partial\mathbb{D}_r(1)\cup\widehat{\gamma}_U\cup\widehat{\gamma}_L$ is displayed in Figure 10.*

(2) *The limiting values $\mathbf{R}_\pm(\lambda),\lambda\in\Sigma_{\mathbf{R}}$ obey*

$$\mathbf{R}_+(\lambda)=\mathbf{R}_-(\lambda)\begin{bmatrix}1 & \gamma\mathrm{e}^{v}\mathrm{e}^{2\mathrm{i}t\lambda}(\frac{\lambda+1}{\lambda-1})^{2\nu}\\ 0 & 1\end{bmatrix},\ \lambda\in\widehat{\gamma}_U;$$

$$\mathbf{R}_+(\lambda)=\mathbf{R}_-(\lambda)\begin{bmatrix}1 & 0\\ -\gamma\mathrm{e}^{v}\mathrm{e}^{-2\mathrm{i}t\lambda}(\frac{\lambda+1}{\lambda-1})^{-2\nu} & 1\end{bmatrix},\ \lambda\in\widehat{\gamma}_L.$$

on the lens boundaries and

$$\mathbf{R}_+(\lambda)=\mathbf{R}_-(\lambda)\mathbf{P}^{(\pm1)}(\lambda)\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1},\ \lambda\in\partial\mathbb{D}_r(\pm1)$$

on the circles.

(3) *As $\lambda\to\infty$, we have $\mathbf{R}(\lambda)\to\mathbb{I}$.*

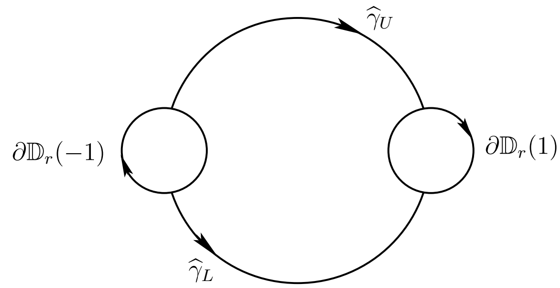

FIGURE 10. The oriented jump contours for the ratio function $\mathbf{R}(\lambda;t,v)$ in the complex λ-plane.

We now argue that the last RHP is asymptotically solvable in the scaling region (5.4) by deriving small norm estimates for the underlying jump matrix $\mathbf{G}_{\mathbf{R}}(\lambda;t,v)$ and using the general theory of [26]. First, from property (2) in RHP 5.2 and (5.5),

Proposition 5.3. *There exists $t_0 > 0$ such that*

$$\|\mathbf{G_R}(\cdot;t,v) - \mathbb{I}\|_{L^2\cap L^\infty(\widehat{\gamma}_U\cup\widehat{\gamma}_L)} \leq \mathrm{e}^{v-2tr-v\alpha(r)}, \qquad \forall\, t \geq t_0, \quad v \in [0,+\infty)$$

where $\alpha(r) = \frac{1}{2}(1 - \frac{2}{\pi}\arctan(\frac{r}{2}))$ and $0 < r < \frac{1}{4}$ is fixed.

Second, using property (2) again and the matching relations (2.5), (2.7) in [17],

Proposition 5.4. *For any fixed $v \in [0,+\infty)$ there exist positive constants $t_0 = t_0(v)$ and $c = c(v)$ such that*

$$\|\mathbf{G_R}(\cdot;t,v) - \mathbb{I}\|_{L^2\cap L^\infty(\partial\mathbb{D}_r(\pm 1))} \leq \frac{c}{t}, \qquad \forall\, t \geq t_0$$

where $0 < r < \frac{1}{4}$ is fixed throughout.

Now combining these two estimates and using [26], we arrive at

Theorem 5.5. *For any fixed $v \in [0,+\infty)$ there exist $t_0 = t_0(v) > 0$ and $c = c(v) > 0$ such that the ratio RHP 5.2 is uniquely solvable in $L^2(\Sigma_\mathbf{R})$ for all $t \geq t_0$. The solution can be computed iteratively through the integral equation*

$$\mathbf{R}(\lambda) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_{\Sigma_\mathbf{R}} \mathbf{R}_-(w)\big(\mathbf{G_R}(w) - \mathbb{I}\big)\frac{\mathrm{d}w}{w-\lambda}, \quad \lambda \in \mathbb{C}\backslash\Sigma_\mathbf{R}$$

with the help of

$$\|\mathbf{R}_-(\cdot;t,v) - \mathbb{I}\|_{L^2(\Sigma_\mathbf{R})} \leq c\,t^{-1}, \qquad \forall\, t \geq t_0.$$

It is now time to extract the relevant asymptotic expansions and substitute the information back into (5.2) and (5.3).

5.2. **Extraction of asymptotics and proof of expansion (1.33).** From the transformation sequence

$$\mathbf{Y}(\lambda) \mapsto \mathbf{X}(\lambda) \mapsto \mathbf{R}(\lambda),$$

we obtain at once the exact identity

$$\mathbf{Y}_1 = 2\nu\sigma_3 + \frac{\mathrm{i}}{2\pi}\int_{\Sigma_\mathbf{R}} \mathbf{R}_-(w)\big(\mathbf{G_R}(w) - \mathbb{I}\big)\,\mathrm{d}w.$$

Now let

$$\mathbf{M} = \frac{\mathrm{i}}{2\pi}\int_{\Sigma_\mathbf{R}} \big(\mathbf{G_R}(w) - \mathbb{I}\big)\,\mathrm{d}w, \qquad \mathbf{M} = \big(M^{jk}\big)_{j,k=1}^2,$$

so that from an explicit residue computation (using Theorem 5.5 and [17], (2.5), (2.7)),

Proposition 5.6. *As $t \to +\infty$,*

$$M^{11} = -\frac{\mathrm{i}\nu^2}{t} + \mathcal{O}\left(t^{-2}\right), \quad M^{12} = \frac{\nu}{t}\sin\left(\varphi(t,v)\right) + \mathcal{O}\left(t^{-2}\right),$$

where

$$\varphi(t,v) = 2t - \frac{v}{\pi}\ln(4t) + 2\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right).$$

All error terms are uniform with respect to v chosen from compact subsets of $[0,+\infty)$.

Thus in turn with (5.2) and (5.3),

Corollary 5.7. *As $t \to +\infty$, with $k \in \mathbb{Z}$ and fixed $v \in [0,+\infty)$,*

$$q(t;\gamma) = -\frac{\mathrm{i}v}{2\pi}\left(1 - \frac{v}{4\pi t} + \frac{v}{2\pi t}\sin^2\left(\varphi(t,v)\right) + \mathcal{O}\left(t^{-2}\right)\right); \qquad p(t;\gamma) = 2\mathrm{i}\varphi(t,v) + 2\pi\mathrm{i}(1+2k) + \mathcal{O}\left(t^{-1}\right)$$

and

$$\mathcal{H}_B\big(q(t;\gamma), p(t;\gamma), t\big) = -\frac{2v}{\pi} + \frac{v^2}{2\pi^2 t} + \mathcal{O}\left(t^{-2}\right).$$

The last result, together with Proposition 1.3, leads us to

Corollary 5.8. *As $t \to +\infty$,*

$$\ln F_B(t;\gamma) = -\frac{2v}{\pi}t + \frac{v^2}{2\pi^2}\ln t + E(v) + \mathcal{O}\left(t^{-1}\right),$$

where $E(v)$ is t-independent and the error term uniform for v chosen from compact subsets of $[0,+\infty)$.

Similar to the last section we now determine $E(v)$ through (1.26) and Corollary 1.6. First, as $t \to +\infty$,

$$t\,\mathcal{H}_B(q,p,t) - pq = -\frac{4vt}{\pi} + \frac{v^2}{\pi^2} - \frac{v^2}{\pi^2}\sin^2\big(\varphi(t,v)\big) + \frac{v^2}{\pi^2}\ln(4t) - \frac{2v}{\pi}\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right) - v(1+2k) + \mathcal{O}\left(t^{-1}\right),$$

with $k \in \mathbb{Z}$ as in Corollary 5.7. But using Corollary 5.7 again it is also easy to see that

$$pq_\gamma = \frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\frac{2vt}{\pi} - \frac{v^2}{2\pi^2} + \frac{v^2}{\pi^2}\sin^2\big(\varphi(t,v)\big) - \frac{v^2}{2\pi^2}\ln(4t) + v(1+2k)\right) + \frac{2v_\gamma}{\pi}\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right) + \mathcal{O}\left(t^{-1}\ln t\right),$$

and therefore together with Corollary 5.8, after one final integration by parts (and the fact $F_B(t;0)=1$),

Proposition 5.9. *As $t \to +\infty$,*

$$\ln F_B(t;\gamma) = -\frac{2v}{\pi}t + \frac{v^2}{2\pi^2}\ln(4t) + \frac{v^2}{2\pi^2} - \frac{2}{\pi}\int_0^\gamma v(\gamma')\frac{\mathrm{d}}{\mathrm{d}\gamma'}\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right)\mathrm{d}\gamma' + \mathcal{O}\left(t^{-1}\right)$$

uniformly for $\gamma \in [0,1)$ chosen from compact subsets.

By standard properties of the Barnes G-function, see (4.7), this results proves Theorem 1.7, expansion (1.33).

6. Proof of Theorem 1.7, expansion (1.35)

As shown in Appendix A below, we can characterize the functions (q,p) in (1.13), (1.19) through the solution of the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 6.1. *Let $t > 0, \alpha > -1$ and $\gamma \in [0,1]$. Determine the piecewise analytic function $\mathbf{Y} = \mathbf{Y}(\lambda;t,\alpha,\gamma) \in \mathbb{C}^{2\times 2}$ such that*

(1) *$\mathbf{Y}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[0,1]$ with the line segment $[0,1] \subset \mathbb{R}$ oriented from left to right.*
(2) *The limiting values $\mathbf{Y}_\pm(\lambda) = \lim_{\epsilon\downarrow 0}\mathbf{Y}(\lambda \pm \mathrm{i}\epsilon)$ along $\lambda \in (0,1)$ obey the jump relation*

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\begin{bmatrix} 1 - \mathrm{i}\pi\gamma\sqrt{\lambda t}J_\alpha(\sqrt{\lambda t})J_\alpha'(\sqrt{\lambda t}) & \mathrm{i}\pi\gamma\big(J_\alpha(\sqrt{\lambda t})\big)^2 \\ -\mathrm{i}\pi\gamma\big(\sqrt{\lambda t}J_\alpha'(\sqrt{\lambda t})\big)^2 & 1 + \mathrm{i}\pi\gamma\sqrt{\lambda t}J_\alpha(\sqrt{\lambda t})J_\alpha'(\sqrt{\lambda t}) \end{bmatrix}, \quad \lambda \in (0,1).$$

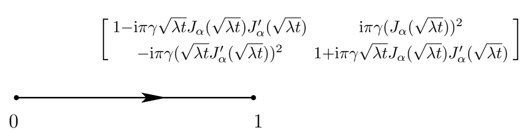

Figure 11. The oriented jump contour for the master function $\mathbf{Y}(\lambda;t,\alpha,\gamma)$ of RHP 6.1 in the complex λ-plane.

(3) *$\mathbf{Y}(\lambda)$ is square integrable on $[0,1] \subset \mathbb{R}$.*
(4) *As $\lambda \to \infty$,*

$$\mathbf{Y}(\lambda) = \mathbb{I} + \mathbf{Y}_1\lambda^{-1} + \mathbf{Y}_2\lambda^{-2} + \mathcal{O}\left(\lambda^{-3}\right), \qquad \mathbf{Y}_\ell = \big(Y_\ell^{jk}\big)_{j,k=1}^2. \tag{6.1}$$

We will prove below that the above problem for $\mathbf{Y}(\lambda)$ is uniquely solvable for all $t \geq t_0$ and $\gamma \in [0,1), \alpha > -1$ fixed. In turn we have the representation formulæ (see Appendix A below)

$$q^2(t,\alpha;\gamma) = t\big(Y_1^{12}\big)^2 + 2\big(Y_1^{11} - Y_1^{12}\big), \qquad p^2(t,\alpha;\gamma) = \frac{\alpha^2q^2}{(q^2-1)^2} + \frac{2t}{q^2-1}\left(Y_1^{12} + \frac{q^2}{2}\right), \tag{6.2}$$

and

$$\mathcal{H}_H\big(q(t,\alpha;\gamma),p(t,\alpha,\gamma),t,\alpha\big)=\frac{1}{2}Y_1^{12}, \tag{6.3}$$

through RHP 6.1. Moreover,

$$L(t,\alpha;\gamma) \;=\; -\frac{\alpha}{2}\ln\Big(-\widehat{X}^{11}(0)\big(\widehat{X}^{12}(0)\big)^{-1}\Big)\Big|_{s=0}^{t}, \qquad -1<\alpha<0, \tag{6.4}$$

$$L(t,\alpha;\gamma) \;=\; -\frac{\alpha}{2}\ln\Big(-\widehat{X}^{11}(0)\big(\widehat{X}^{12}(0)\big)^{-1}s^{-\alpha}\Big)\Big|_{s=0}^{t}, \qquad \alpha\geq 0. \tag{6.5}$$

in terms of RHP 6.2 below. Formulæ (6.2), (6.3) and (6.4), (6.5) are the starting point for our asymptotic analysis, but the necessary nonlinear steepest descent techniques (for $\gamma\in[0,1)$ that is, in case $\gamma=1$ see [15], Section 6) have not appeared in the literature yet, thus we provide the details below.

6.1. **Nonlinear steepest descent analysis for RHP 6.1.** Our goal is to solve RHP 6.1 for $\mathbf{Y}(\lambda;t,\alpha,\gamma)\in\mathbb{C}^{2\times 2}$ for all values $(t,\alpha,v)\in\mathbb{R}_+\times\mathbb{R}_{>-1}\times\mathbb{R}_+$ such that

$$t\geq t_0,\quad \text{and}\quad \alpha>-1,\quad 0\leq v=-\ln(1-\gamma)<+\infty\quad \text{are fixed.}$$

To this end we shall first recall a few key steps from [15], Section 6.1. Let $\Psi_\alpha(\zeta)$ denote the function defined in (B.1) below. It allows us to factorize the jump matrix in RHP 6.1 as follows, for $\lambda>0$,

$$\big(\Psi_\alpha(\lambda t)\big)_+^{-1}\begin{bmatrix}1-\mathrm{i}\pi\gamma\sqrt{\lambda t}J_\alpha(\sqrt{\lambda t})J_\alpha'(\sqrt{\lambda t}) & \mathrm{i}\pi\gamma\big(J_\alpha(\sqrt{\lambda t})\big)^2\\ -\mathrm{i}\pi\gamma\big(\sqrt{\lambda t}J_\alpha'(\sqrt{\lambda t})\big)^2 & 1+\mathrm{i}\pi\gamma\sqrt{\lambda t}J_\alpha(\sqrt{\lambda t})J_\alpha'(\sqrt{\lambda t})\end{bmatrix}\big(\Psi_\alpha(\lambda t)\big)_+=\begin{bmatrix}1&-\gamma\\0&1\end{bmatrix}, \tag{6.6}$$

and thus motivates an undressing transformation. In more detail, using the model function $\Psi(\zeta;\alpha)$ from (B.2) and RHP B.1 from Appendix B, we define

$$\mathbf{X}(\lambda)=\begin{bmatrix}1&0\\b_1(\alpha)&1\end{bmatrix}\mathbf{Y}(\lambda)\Psi(\lambda t;\alpha)\begin{cases}\left[\begin{smallmatrix}1&0\\ \mathrm{e}^{-\mathrm{i}\pi\alpha}&1\end{smallmatrix}\right], & \lambda\in\widehat{\Omega}_1\\ \left[\begin{smallmatrix}1&0\\ -\mathrm{e}^{\mathrm{i}\pi\alpha}&1\end{smallmatrix}\right], & \lambda\in\widehat{\Omega}_2\\ \mathbb{I}, & \text{else}\end{cases}. \tag{6.7}$$

In view of Figure 12, this step leads us to the following transformed RHP.

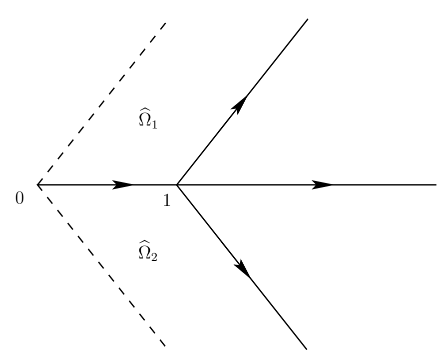

FIGURE 12. The oriented jump contours as solid black lines for the function $\mathbf{X}(\lambda)$ defined in (6.7) in the complex λ-plane.

Riemann-Hilbert Problem 6.2. *Find $\mathbf{X}(\lambda)=\mathbf{X}(\lambda;t,\alpha,v)\in\mathbb{C}^{2\times 2}$ with $(t,\alpha,v)\in\mathbb{R}_{>0}\times\mathbb{R}_{>-1}\times\mathbb{R}_{\geq 0}$ such that*

(1) *$\mathbf{X}(\lambda)$ is analytic for $\lambda\in\mathbb{C}\backslash\Sigma_{\mathbf{X}}$ where the oriented contour $\Sigma_{\mathbf{X}}$ is shown in Figure 12 as union of solid black lines.*

(2) *The jumps on $\Sigma_{\mathbf{X}}$ read as*

$$\mathbf{X}_+(\lambda)=\mathbf{X}_-(\lambda)\begin{bmatrix}\mathrm{e}^{-\mathrm{i}\pi\alpha} & \mathrm{e}^{-v}\\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha}\end{bmatrix},\quad \lambda\in(0,1);\qquad \mathbf{X}_+(\lambda)=\mathbf{X}_-(\lambda)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \lambda\in(1,+\infty)$$

on the positive real axis and

$$\begin{aligned}\mathbf{X}_+(\lambda)&=\mathbf{X}_-(\lambda)\begin{bmatrix}1 & 0\\ \mathrm{e}^{-\mathrm{i}\pi\alpha} & 1\end{bmatrix},\quad \arg(\lambda-1)=\frac{\pi}{3};\\ \mathbf{X}_+(\lambda)&=\mathbf{X}_-(\lambda)\begin{bmatrix}1 & 0\\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{bmatrix},\quad \arg(\lambda-1)=\frac{5\pi}{3}.\end{aligned}$$

(3) *In a vicinity of $\lambda=0$,*

$$\mathbf{X}(\lambda)=\widehat{\mathbf{X}}(\lambda)(-\lambda)^{\frac{\alpha}{2}\sigma_3}\begin{cases}\begin{bmatrix}1 & \frac{\mathrm{i}}{2}\frac{1-\gamma}{\sin\pi\alpha}\\ 0 & 1\end{bmatrix}, & \alpha\notin\mathbb{Z}\\ \begin{bmatrix}1 & -\frac{\mathrm{e}^{\mathrm{i}\pi\alpha}}{2\pi\mathrm{i}}\mathrm{e}^{-v}\ln(-\lambda)\\ 0 & 1\end{bmatrix}, & \alpha\in\mathbb{Z}\end{cases}$$

where $\ln:\mathbb{C}\backslash(-\infty,0]\to\mathbb{C}$ *and* $z^\alpha:\mathbb{C}\backslash(-\infty,0]\to\mathbb{C}$ *are defined with principal branches and* $\widehat{\mathbf{X}}(\lambda)=\big(\widehat{X}^{jk}(\lambda)\big)_{j,k=1}^2$ *is analytic at $\lambda=0$.*

(4) *In a vicinity of $\lambda=1$,*

$$\mathbf{X}(\lambda)=\widehat{\mathbf{X}}(\lambda)\left\{\mathbb{I}+\frac{\gamma}{2\pi\mathrm{i}}\begin{bmatrix}-1 & -\mathrm{e}^{-\mathrm{i}\pi\alpha}\\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{bmatrix}\ln(\lambda-1)\right\}\begin{cases}\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right], & \Im\lambda>0\\ \mathbb{I}, & \Im\lambda<0\end{cases}$$
$$\times\begin{cases}\left[\begin{smallmatrix}1 & 0\\ \mathrm{e}^{-\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg(\lambda-1)\in(\frac{\pi}{3},\pi)\\ \left[\begin{smallmatrix}1 & 0\\ -\mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg(\lambda-1)\in(\pi,\frac{5\pi}{3})\\ \mathbb{I}, & \text{else}\end{cases},$$

with $\widehat{\mathbf{X}}(\lambda)$ analytic at $\lambda=1$ and the principal branch for $\ln(\lambda-1)$*, i.e.* $-\pi<\arg(\lambda-1)<\pi$*.*

(5) *Using RHP B.1 we find that as $\lambda\to\infty,\lambda\notin[0,+\infty)$,*

$$\mathbf{X}(\lambda)=\Big\{\mathbb{I}+\mathbf{X}_1\lambda^{-1}+\mathcal{O}\left(\lambda^{-2}\right)\Big\}(-\lambda t)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sigma_3}\mathrm{e}^{\sqrt{t}(-\lambda)^{\frac{1}{2}}\sigma_3}$$

with

$$\mathbf{X}_1=\begin{bmatrix}1 & 0\\ b_1(\alpha) & 1\end{bmatrix}\mathbf{Y}_1\begin{bmatrix}1 & 0\\ -b_1(\alpha) & 1\end{bmatrix}-\frac{1}{t}\begin{bmatrix}a_2(\alpha) & -a_1(\alpha)\\ b_1(\alpha)a_2(\alpha)-b_3(\alpha) & b_2(\alpha)-b_1(\alpha)a_1(\alpha)\end{bmatrix}.$$

Our next step is the g-function transformation given by

$$\mathbf{T}(\lambda)=\mathbf{X}(\lambda)\mathrm{e}^{-\sqrt{t}g(\lambda)\sigma_3},\quad \lambda\in\mathbb{C}\backslash\Sigma_{\mathbf{X}};\qquad g(\lambda)=(-\lambda)^{\frac{1}{2}},\quad \lambda\in\mathbb{C}\backslash[0,\infty)\tag{6.8}$$

where $g(\lambda)$ is defined and analytic for $\lambda\in\mathbb{C}\backslash[0,+\infty)$ such that $(-\lambda)^{\frac{1}{2}}>0$ for $\lambda<0$.

Proposition 6.3. *We have*

$$g_\pm(\lambda)=\sqrt{|\lambda|},\quad \lambda<0;\qquad g_\pm(\lambda)=\mp\mathrm{i}\sqrt{\lambda},\quad \lambda>0;\qquad \Re\big(g(\lambda)\big)>0,\quad \arg(\lambda-1)=\frac{\pi}{3},\frac{5\pi}{3}$$

and $\Pi(\lambda)=2\mathrm{i}\sqrt{\lambda},\lambda>0$ admits analytic continuation into a small vicinity of $(0,1)$ into the lower and upper half plane. In fact with

$$\phi(\lambda)=-2(-\lambda)^{\frac{1}{2}},\qquad \lambda\in\mathbb{C}\backslash[0,\infty)$$

we observe that

$$\phi_+(\lambda)=\Pi(\lambda)=-\phi_-(\lambda),\quad \lambda>0;\qquad \Re\big(\phi(\lambda)\big)<0\quad\text{for }\Im\lambda\gtrless 0,\quad \Re\lambda\in(0,1).$$

Recalling RHP 6.2 the transformation (6.8) leads to the following problem

Riemann-Hilbert Problem 6.4. *Find $\mathbf{T}(\lambda)=\mathbf{T}(\lambda;t,\alpha,v)\in\mathbb{C}^{2\times2}$ with $(t,\alpha,v)\in\mathbb{R}_{>0}\times\mathbb{R}_{>-1}\times\mathbb{R}_{\geq0}$ such that*

(1) $\mathbf{T}(\lambda)$ *is analytic for* $\lambda \in \mathbb{C}\backslash\Sigma_{\mathbf{X}}$ *with* $\Sigma_{\mathbf{X}}$ *shown in Figure 12.*
(2) *On the contour* $\Sigma_{\mathbf{X}}$,

$$\mathbf{T}_+(\lambda) = \mathbf{T}_-(\lambda)\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha}\mathrm{e}^{\sqrt{t}\,\Pi(\lambda)} & 1-\gamma \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha}\mathrm{e}^{-\sqrt{t}\,\Pi(\lambda)} \end{bmatrix}, \quad \lambda \in (0,1); \qquad \mathbf{T}_+(\lambda) = \mathbf{T}_-(\lambda)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \lambda \in (1,+\infty)$$

followed by

$$\mathbf{T}_+(\lambda) = \mathbf{T}_-(\lambda)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-\mathrm{i}\pi\alpha}\mathrm{e}^{-2\sqrt{t}\,g(\lambda)} & 1 \end{bmatrix}, \quad \arg(\lambda-1) = \frac{\pi}{3},$$

$$\mathbf{T}_+(\lambda) = \mathbf{T}_-(\lambda)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\mathrm{i}\pi\alpha}\mathrm{e}^{-2\sqrt{t}\,g(\lambda)} & 1 \end{bmatrix}, \quad \arg(\lambda-1) = \frac{5\pi}{3}.$$

(3) *The singular behavior near* $\lambda = 0$ *and* $\lambda = 1$ *is, modulo the right multiplication with the g-function, see (6.8), unchanged from the corresponding behavior stated in RHP 6.2, compare conditions (3) and (4).*
(4) *As* $\lambda \to \infty, \lambda \notin [0,+\infty)$, *we have that*

$$\mathbf{T}(\lambda) = \Big\{\mathbb{I} + \mathbf{X}_1\lambda^{-1} + \mathcal{O}\left(\lambda^{-2}\right)\Big\}(-\lambda t)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sigma_3}.$$

Since

$$\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha}\mathrm{e}^{\sqrt{t}\,\Pi(\lambda)} & 1-\gamma \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha}\mathrm{e}^{-\sqrt{t}\,\Pi(\lambda)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi_-(\lambda)}\mathrm{e}^{v+\mathrm{i}\pi\alpha} & 1 \end{bmatrix}\begin{bmatrix} 0 & \mathrm{e}^{-v} \\ -\mathrm{e}^{v} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi_+(\lambda)}\mathrm{e}^{v-\mathrm{i}\pi\alpha} & 1 \end{bmatrix},$$

we can use Proposition 6.3 and perform our next transformation. Define with the help of Figure 13

$$\mathbf{S}(\lambda) = \mathbf{T}(\lambda)\begin{cases} \begin{bmatrix} 1 & 0 \\ -\mathrm{e}^{\sqrt{t}\,\phi(\lambda)}\mathrm{e}^{v-\mathrm{i}\pi\alpha} & 1 \end{bmatrix}, & \lambda \in \Omega_U \\ \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)}\mathrm{e}^{v+\mathrm{i}\pi\alpha} & 1 \end{bmatrix}, & \lambda \in \Omega_L \\ \mathbb{I}, & \text{else} \end{cases} \tag{6.9}$$

and obtain the following problem.

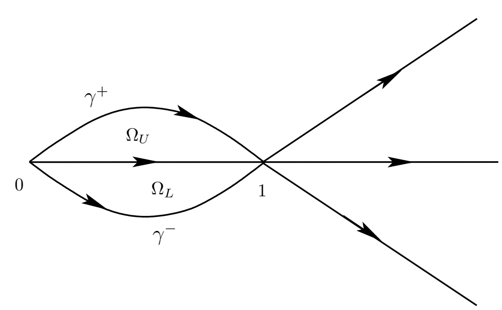

FIGURE 13. The domains used in Definition (6.9). The union of the solid black contours equals $\Sigma_{\mathbf{S}}$.

Riemann-Hilbert Problem 6.5. *Find* $\mathbf{S}(\lambda) = \mathbf{S}(\lambda; t, \alpha, v) \in \mathbb{C}^{2\times 2}$ *such that*

(1) $\mathbf{S}(\lambda)$ *is analytic for* $\lambda \in \mathbb{C}\backslash\Sigma_{\mathbf{S}}$ *and* $\Sigma_{\mathbf{S}}$ *is shown in Figure 13.*
(2) *The jumps are as follows,*

$$\mathbf{S}_+(\lambda) = \mathbf{S}_-(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathrm{e}^{\frac{v}{2}\sigma_3}, \quad \lambda \in (0,1); \qquad \mathbf{S}_+(\lambda) = \mathbf{S}_-(\lambda)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \lambda \in (1,+\infty);$$

$$\mathbf{S}_+(\lambda) = \mathbf{S}_-(\lambda)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)}\mathrm{e}^{v-\mathrm{i}\pi\alpha} & 1 \end{bmatrix}, \quad \lambda \in \gamma^+; \quad \mathbf{S}_+(\lambda) = \mathbf{S}_-(\lambda)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)}\mathrm{e}^{v+\mathrm{i}\pi\alpha} & 1 \end{bmatrix}, \quad \lambda \in \gamma^-;$$

$$\mathbf{S}_+(\lambda)=\mathbf{S}_-(\lambda)\begin{bmatrix}1 & 0\\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)-\mathrm{i}\pi\alpha} & 1\end{bmatrix},\quad \arg(\lambda-1)=\frac{\pi}{3};$$
$$\mathbf{S}_+(\lambda)=\mathbf{S}_-(\lambda)\begin{bmatrix}1 & 0\\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)+\mathrm{i}\pi\alpha} & 1\end{bmatrix},\quad \arg(\lambda-1)=\frac{5\pi}{3}.$$

(3) *The singular behavior of $\mathbf{T}(\lambda)$ near $\lambda=0$ and $\lambda=1$ has to be adjusted according to (6.9), i.e. we have to multiply the local expansions by the appropriate right multipliers.*

(4) *The behavior near $\lambda=\infty$ remains unchanged from RHP 6.4, i.e.*

$$\mathbf{S}(\lambda)=\Big\{\mathbb{I}+\mathbf{X}_1\lambda^{-1}+\mathcal{O}\left(\lambda^{-2}\right)\Big\}(-\lambda t)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sigma_3},\qquad \lambda\to\infty,\quad \lambda\notin[0,+\infty).\tag{6.10}$$

We have now reached the point where the problem is localized. Indeed, in view of Proposition 6.3, we have for the jump matrix $\mathbf{G}_{\mathbf{S}}(\lambda;t,\alpha,v)$ away from $\lambda=0,1$,

$$\mathbf{G}_{\mathbf{S}}(\lambda;t,\alpha,v)=\mathbb{I}+\mathcal{O}\left(\mathrm{e}^{v-c\sqrt{t|\lambda|}}\right),\quad c>0,\quad \lambda\in\gamma^+\cup\gamma^-\tag{6.11}$$

and

$$\mathbf{G}_{\mathbf{S}}(\lambda;t,\alpha,v)=\mathbb{I}+\mathcal{O}\left(\mathrm{e}^{-d\sqrt{t}}\right),\quad d>0,\quad \arg(\lambda-1)=\frac{\pi}{3},\frac{5\pi}{3}.\tag{6.12}$$

For this reason we now focus on the local analysis on $(0,+\infty)\subset\mathbb{R}$ and near $\lambda=0,1$. First, the parametrix $\mathbf{P}^{(\infty)}(\lambda)$ for the line segment $(0,\infty)\subset\mathbb{R}$ will obey the following conditions:

Riemann-Hilbert Problem 6.6. *Determine $\mathbf{P}^{(\infty)}(\lambda)=\mathbf{P}^{(\infty)}(\lambda;t,v)\in\mathbb{C}^{2\times2}$ such that*

(1) *$\mathbf{P}^{(\infty)}(\lambda)$ is analytic for $\lambda\in\mathbb{C}\backslash[0,\infty)$*

(2) *We require that $\mathbf{P}^{(\infty)}(\lambda)$ assumes square integrable boundary values on $[0,\infty)$ which satisfy the jump conditions*

$$\mathbf{P}^{(\infty)}_+(\lambda)=\mathbf{P}^{(\infty)}_-(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathrm{e}^{\frac{v}{2}\sigma_3},\quad \lambda\in(0,1);$$

and

$$\mathbf{P}^{(\infty)}_+(\lambda)=\mathbf{P}^{(\infty)}_-(\lambda)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \lambda\in(1,\infty).$$

(3) *As $\lambda\to\infty$ with $\lambda\notin[0,\infty)\subset\mathbb{R}$, see (6.10),*

$$\mathbf{P}^{(\infty)}(\lambda)=\left\{\mathbb{I}+\frac{1}{\lambda}\begin{bmatrix}2\nu^2 & -\frac{2\mathrm{i}\nu}{\sqrt{t}}\\ \frac{2\mathrm{i}}{3}\nu(1-4\nu^2)\sqrt{t} & -2\nu^2\end{bmatrix}+\mathcal{O}\left(\lambda^{-2}\right)\right\}(-\lambda t)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sigma_3}.$$

It is easy to check that

$$\mathbf{P}^{(\infty)}(\lambda)=\begin{bmatrix}1 & 0\\ -2\mathrm{i}\nu\sqrt{t} & 1\end{bmatrix}(-\lambda t)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sigma_3}\big(\mathcal{D}(\lambda)\big)^{-\sigma_3},\qquad \lambda\in\mathbb{C}\backslash[0,\infty)\tag{6.13}$$

with

$$\mathcal{D}(\lambda)=\exp\left[(-\lambda)^{\frac{1}{2}}\frac{v}{2\pi}\int_0^1\frac{\mathrm{d}w}{\sqrt{w}\,(w-\lambda)}\right]=\left(\frac{(-\lambda)^{\frac{1}{2}}-\mathrm{i}}{(-\lambda)^{\frac{1}{2}}+\mathrm{i}}\right)^{\nu},\qquad \nu=\frac{v}{2\pi\mathrm{i}}\in\mathrm{i}\mathbb{R}\tag{6.14}$$

solves the above problem, provided we choose principal branches for all fractional exponents in (6.13) and (6.14). Next, for the parametrix $\mathbf{P}^{(0)}(\lambda)$ in a vicinity of $\lambda=0$, we require the following properties:

Riemann-Hilbert Problem 6.7. *Determine $\mathbf{P}^{(0)}(\lambda)=\mathbf{P}^{(0)}(\lambda;t,\alpha,v)\in\mathbb{C}^{2\times2}$ such that*

(1) *$\mathbf{P}^{(0)}(\lambda)$ is analytic for $\lambda\in\mathbb{D}_{\frac{1}{4}}(0)\backslash(\Sigma_{\mathbf{S}}\cup\{0\})$ with $\mathbb{D}_r(\lambda_0)=\{\lambda\in\mathbb{C}:\ |\lambda-\lambda_0|<r\}$.*

(2) *On the three contours near $\lambda=0$ (compare Figure 13), the function $\mathbf{P}^{(0)}(\lambda)$ behaves as follows,*

$$\begin{aligned}\mathbf{P}^{(0)}_+(\lambda)&=\mathbf{P}^{(0)}_-(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathrm{e}^{\frac{v}{2}\sigma_3},\qquad \lambda\in(0,1)\cap\mathbb{D}_{\frac{1}{4}}(0);\\ \mathbf{P}^{(0)}_+(\lambda)&=\mathbf{P}^{(0)}_-(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3}\begin{bmatrix}1 & 0\\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)-\mathrm{i}\pi\alpha} & 1\end{bmatrix}\mathrm{e}^{\frac{v}{2}\sigma_3},\qquad \lambda\in\gamma^+\cap\mathbb{D}_{\frac{1}{4}}(0);\\ \mathbf{P}^{(0)}_+(\lambda)&=\mathbf{P}^{(0)}_-(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3}\begin{bmatrix}1 & 0\\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)+\mathrm{i}\pi\alpha} & 1\end{bmatrix}\mathrm{e}^{\frac{v}{2}\sigma_3},\qquad \lambda\in\gamma^-\cap\mathbb{D}_{\frac{1}{4}}(0).\end{aligned}$$

(3) *As $\lambda \to 0$, the singular behavior of $\mathbf{P}^{(0)}(\lambda)$ matches the behavior of the function $\mathbf{S}(\lambda)$ as described in RHP 6.5, condition (3).*

(4) *As $t \to +\infty$ with $\gamma \in [0,1)$ fixed, the local functions $\mathbf{P}^{(\infty)}(\lambda)$ and $\mathbf{P}^{(0)}(\lambda)$ obey the matching*

$$\mathbf{P}^{(0)}(\lambda) \sim \left\{ \mathbb{I} + \sum_{m=1}^{\infty} \mathbf{E}^{(0)}(\lambda)\mathcal{S}_m(\alpha)\big(\mathbf{E}^{(0)}(\lambda)\big)^{-1}(-\lambda t)^{-m} \right\} \mathbf{P}^{(\infty)}(\lambda),$$

with (compare RHP B.1 in Appendix B)

$$\mathcal{S}_m(\alpha) = \begin{bmatrix} a_{2m}(\alpha) & -a_{2m-1}(\alpha) \\ b_1(\alpha)a_{2m}(\alpha) - b_{2m+1}(\alpha) & b_{2m}(\alpha) - b_1(\alpha)a_{2m-1}(\alpha) \end{bmatrix}$$

which holds uniformly for $0 < r_1 \leq |\lambda| \leq r_2 < \frac{1}{4}$ where r_1 and r_2 are fixed. The multiplier $\mathbf{E}^{(0)}(\lambda)$ is defined in (6.16) below.

A solution to this problem is most easily constructed by recalling RHP B.1, or equivalently (B.2): we define

$$\mathbf{P}^{(0)}(\lambda) = \mathbf{E}^{(0)}(\lambda) \begin{bmatrix} 1 & 0 \\ b_1(\alpha) & 1 \end{bmatrix} \Psi(\lambda t; \alpha) \mathrm{e}^{-\sqrt{t}\, g(\lambda)\sigma_3} \mathrm{e}^{\frac{v}{2}\sigma_3}, \qquad \lambda \in \mathbb{D}_{\frac{1}{4}}(0) \setminus (\Sigma_{\mathbf{S}} \cup \{0\}) \tag{6.15}$$

using the locally analytic multiplier

$$\mathbf{E}^{(0)}(\lambda) = \mathbf{P}^{(\infty)}(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}(-\lambda t)^{\frac{1}{4}\sigma_3}, \qquad \lambda \in \mathbb{D}_{\frac{1}{4}}(0). \tag{6.16}$$

Remark 6.8. *As $\lambda \to 0$ we have*

$$\mathbf{E}^{(0)}(\lambda) = t^{-\frac{1}{4}\sigma_3} \begin{bmatrix} 1 & -2\mathrm{i}\nu \\ -2\mathrm{i}\nu & 1 - 4\nu^2 \end{bmatrix} \left\{ \mathbb{I} + \lambda \begin{bmatrix} -2\nu^2 & -\frac{2\mathrm{i}}{3}\nu(1-4\nu^2) \\ 2\mathrm{i}\nu & 2\nu^2 \end{bmatrix} + \mathcal{O}\left(\lambda^2\right) \right\} t^{\frac{1}{4}\sigma_3}$$

and from (6.13) we see that $\mathbf{E}^{(0)}(\lambda) = t^{-\frac{1}{4}\sigma_3}\widehat{\mathbf{E}}^{(0)}(\lambda)t^{\frac{1}{4}\sigma_3}$ with $\widehat{\mathbf{E}}^{(0)}(\lambda)$ independent of t.

Our final parametrix near $\lambda = 1$ obeys the following conditions:

Riemann-Hilbert Problem 6.9. *Find $\mathbf{P}^{(1)}(\lambda) = \mathbf{P}^{(1)}(\lambda; t, \alpha, v) \in \mathbb{C}^{2\times 2}$ such that*

(1) *$\mathbf{P}^{(1)}(\lambda)$ is analytic for $\lambda \in \mathbb{D}_{\frac{1}{4}}(1) \setminus (\Sigma_{\mathbf{S}} \cup \{1\})$, see Figure 13 for orientations.*

(2) *Along Σ_S, the limiting values $\mathbf{P}_{\pm}^{(1)}(\lambda)$ are square integrable and*

$$\mathbf{P}_+^{(1)}(\lambda) = \mathbf{P}_-^{(1)}(\lambda) \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda) - \mathrm{i}\pi\alpha} & 1 \end{bmatrix}, \qquad \lambda \in \left\{ \lambda \in \mathbb{C} : \ \arg(\lambda - 1) = \frac{\pi}{3} \right\} \cap \mathbb{D}_{\frac{1}{4}}(1);$$

$$\mathbf{P}_+^{(1)}(\lambda) = \mathbf{P}_-^{(1)}(\lambda) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \lambda \in \{ \lambda \in \mathbb{C} : \ \arg(\lambda - 1) = 0 \} \cap \mathbb{D}_{\frac{1}{4}}(1);$$

$$\mathbf{P}_+^{(1)}(\lambda) = \mathbf{P}_-^{(1)}(\lambda) \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda) + \mathrm{i}\pi\alpha} & 1 \end{bmatrix}, \qquad \lambda \in \left\{ \lambda \in \mathbb{C} : \ \arg(\lambda - 1) = \frac{5\pi}{3} \right\} \cap \mathbb{D}_{\frac{1}{4}}(1).$$

Moreover,

$$\mathbf{P}_+^{(1)}(\lambda) = \mathbf{P}_-^{(1)}(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3} \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda) \mp \mathrm{i}\pi\alpha} & 1 \end{bmatrix} \mathrm{e}^{\frac{v}{2}\sigma_3}, \qquad \lambda \in \gamma^{\pm} \cap \mathbb{D}_{\frac{1}{4}}(1);$$

$$\mathbf{P}_+^{(1)}(\lambda) = \mathbf{P}_-^{(1)}(\lambda)\mathrm{e}^{-\frac{v}{2}\sigma_3} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathrm{e}^{\frac{v}{2}\sigma_3}, \qquad \lambda \in \{ \lambda \in \mathbb{C} : \ \arg(\lambda - 1) = \pi \} \cap \mathbb{D}_{\frac{1}{4}}(1).$$

(3) *Near $\lambda = 1$, the singular behavior of $\mathbf{P}^{(1)}(\lambda)$ is exactly of the form given in RHP 6.5, condition (3).*

(4) *As $t \to \infty$ with $\gamma \in [0,1)$ fixed, the two model functions $\mathbf{P}^{(\infty)}(\lambda)$ and $\mathbf{P}^{(1)}(\lambda)$ are related via*

$$\mathbf{P}^{(1)}(\lambda) \sim \left\{ \mathbb{I} + \sum_{m=1}^{\infty} \mathbf{E}^{(1)}(\lambda)\mathcal{U}_m(\nu)\big(\mathbf{E}^{(1)}(\lambda)\big)^{-1}\big(\mathrm{i}\zeta(\lambda)\big)^{-m} \right\} \mathbf{P}^{(\infty)}(\lambda),$$

with (compare RHP B.3 below)

$$\mathcal{U}_m(\nu) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3} \begin{bmatrix} ((\nu)_m)^2 & (-1)^m m((1-\nu)_{m-1})^2 \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ m\,((1+\nu)_{m-1})^2 \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & (-1)^m((-\nu)_m)^2 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\frac{1}{m!}$$

which holds uniformly for $0 < r_1 \leq |\lambda - 1| \leq r_2 < \frac{1}{4}$ with fixed r_1 and r_2. The multiplier $\mathbf{E}^{(1)}(\lambda)$ is defined in (6.19) below and the local change of coordinates $\zeta = \zeta(\lambda)$ in (6.18).

The last problem is solved in terms of the confluent hypergeometric function $U(a, \zeta) \equiv U(a, 1, \zeta)$, see [60], and our construction makes use of the model function $\Phi(\zeta)$ described in (B.4) below, see again Appendix B. In more detail, define

$$\mathbf{P}^{(1)}(\lambda) = \mathbf{E}^{(1)}(\lambda)\Phi\big(\zeta(\lambda)\big)\begin{cases} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\mathrm{e}^{\frac{\mathrm{i}}{2}(\zeta(\lambda)+2\sqrt{t})\sigma_3}, & \lambda \in \mathbb{D}_{\frac{1}{4}}(1)\backslash\Sigma_{\mathbf{S}}: \ \arg\lambda \in (0, \pi) \\ \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\mathrm{e}^{-\frac{\mathrm{i}}{2}(\zeta(\lambda)+2\sqrt{t})\sigma_3}, & \lambda \in \mathbb{D}_{\frac{1}{4}}(1)\backslash\Sigma_{\mathbf{S}}: \ \arg\lambda \in (\pi, 2\pi) \end{cases} \times \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\nu+1)\sigma_3}, \quad (6.17)$$

where

$$\lambda \in \mathbb{D}_{\frac{1}{4}}(1): \quad \zeta(\lambda) = 2\sqrt{t}\big(\mathrm{i}\,\mathrm{sgn}(\Im\lambda)g(\lambda) - 1\big) = \sqrt{t}\,(\lambda - 1)\left(1 - \frac{1}{4}(\lambda - 1) + \mathcal{O}\big((\lambda-1)^2\big)\right), \quad \lambda \to 1, \quad (6.18)$$

and

$$\mathbf{E}^{(1)}(\lambda) = \mathbf{P}^{(\infty)}(\lambda)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)\sigma_3}\begin{cases} \mathrm{e}^{-\mathrm{i}\sqrt{t}\,\sigma_3}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\begin{bmatrix} 0 & \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu} \\ -\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}\nu} & 0 \end{bmatrix}\big(\zeta(\lambda)\big)^{\nu\sigma_3}, & \lambda \in \mathbb{D}_{\frac{1}{4}}(1): \ \arg(\lambda-1) \in (\frac{\pi}{2}, \pi) \\ \mathrm{e}^{-\mathrm{i}\sqrt{t}\,\sigma_3}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\begin{bmatrix} 0 & \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}\nu} \\ -\mathrm{e}^{-\mathrm{i}\frac{7\pi}{2}\nu} & 0 \end{bmatrix}\big(\zeta(\lambda)\big)^{\nu\sigma_3}, & \lambda \in \mathbb{D}_{\frac{1}{4}}(1): \ \arg(\lambda-1) \in (2\pi, \frac{5\pi}{2}) \\ \mathrm{e}^{\mathrm{i}\sqrt{t}\,\sigma_3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\frac{5\pi}{2}\nu} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\frac{3\pi}{2}\nu} \end{bmatrix}\big(\zeta(\lambda)\big)^{\nu\sigma_3}, & \lambda \in \mathbb{D}_{\frac{1}{4}}(1): \ \arg(\lambda-1) \in (\pi, 2\pi) \end{cases} \quad (6.19)$$

are both analytic at $\lambda = 1$ (note that ζ^ν is defined with a cut on the positive imaginary ζ-axis, see (B.3) below). Once we recall RHP B.3 it is easy to verify that (6.17) has all the properties required in RHP 6.9.

Remark 6.10. *Using (6.19) we derive the following Taylor expansion of $\mathbf{E}^{(1)}(\lambda)$ near $\lambda = 1$,*

$$\begin{aligned}\mathbf{E}^{(1)}(\lambda) = {}& t^{-\frac{1}{4}\sigma_3}\begin{bmatrix} 1 & 0 \\ -2\mathrm{i}\nu & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -\mathrm{i} & -1 \\ 1 & \mathrm{i} \end{bmatrix}4^{\nu\sigma_3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)\sigma_3}\mathrm{e}^{\mathrm{i}\sqrt{t}\,\sigma_3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}t^{\frac{\nu}{2}\sigma_3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu} \\ & \times\left\{\mathbb{I} + \frac{1}{4}(\lambda - 1)\begin{bmatrix} \nu & -\mathrm{i}\,16^{-\nu}\mathrm{e}^{\mathrm{i}\pi(\nu+\alpha)}t^{-\nu}\mathrm{e}^{-2\mathrm{i}\sqrt{t}} \\ \mathrm{i}\,16^{\nu}\mathrm{e}^{-\mathrm{i}\pi(\nu+\alpha)}t^{\nu}\mathrm{e}^{2\mathrm{i}\sqrt{t}} & -\nu \end{bmatrix} + \mathcal{O}\big((\lambda-1)^2\big)\right\}, \quad \lambda \to 1.\end{aligned}$$

This concludes the local analysis and we now compare (6.13), (6.15) and (6.17) to the function $\mathbf{S}(\lambda)$. Introduce

$$\mathbf{R}(\lambda) = \mathbf{S}(\lambda)\begin{cases} \big(\mathbf{P}^{(0)}(\lambda)\big)^{-1}, & \lambda \in \mathbb{D}_r(0) \\ \big(\mathbf{P}^{(1)}(\lambda)\big)^{-1}, & \lambda \in \mathbb{D}_r(1) \\ \big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, & \lambda \notin \big(\mathbb{D}_r(0) \cup \mathbb{D}_r(1)\big) \end{cases}, \quad (6.20)$$

where $0 < r < \frac{1}{4}$ is kept fixed. In view of RHP 6.5, 6.6, 6.7 and 6.9, we derive the following RHP for the ratio function (6.20).

Riemann-Hilbert Problem 6.11. *Find $\mathbf{R}(\lambda) = \mathbf{R}(\lambda; t, \alpha, v) \in \mathbb{C}^{2\times 2}$ such that*

(1) *$\mathbf{R}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash\Sigma_{\mathbf{R}}$ and assumes square-integrable boundary values $\mathbf{R}_\pm(\lambda)$ on the oriented contour*

$$\Sigma_{\mathbf{R}} = \partial\mathbb{D}_r(0) \cup \partial\mathbb{D}_r(1) \cup \left(\left(\widehat{\gamma}^+ \cup \widehat{\gamma}^- \cup \left\{\arg(\lambda - 1) = \frac{\pi}{3}, \frac{5\pi}{3}\right\}\right) \cap \big\{\lambda \in \mathbb{C}: \ |\lambda| > r, \ |\lambda - 1| > r\big\}\right)$$

shown in Figure 14 below.

(2) *We have $\mathbf{R}_+(\lambda) = \mathbf{R}_-(\lambda)\mathbf{G}_{\mathbf{R}}(\lambda), \lambda \in \Sigma_{\mathbf{R}}$ where*

$$\mathbf{G}_{\mathbf{R}}(\lambda) = \mathbf{P}^{(0)}(\lambda)\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, \quad \lambda \in \partial\mathbb{D}_r(0); \qquad \mathbf{G}_{\mathbf{R}}(\lambda) = \mathbf{P}^{(1)}(\lambda)\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, \quad \lambda \in \partial\mathbb{D}_r(1),$$

followed by

$$\mathbf{G}_{\mathbf{R}}(\lambda) = \mathbf{P}^{(\infty)}(\lambda)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)}\mathrm{e}^{v\mp\mathrm{i}\pi\alpha} & 1 \end{bmatrix}\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, \quad \lambda \in \widehat{\gamma}^\pm,$$

and concluding with

$$\mathbf{G}_{\mathbf{R}}(\lambda) = \mathbf{P}^{(\infty)}(\lambda)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\sqrt{t}\,\phi(\lambda)\mp\mathrm{i}\pi\alpha} & 1 \end{bmatrix}\big(\mathbf{P}^{(\infty)}(\lambda)\big)^{-1}, \quad \lambda \in \left\{\arg(\lambda - 1) = \frac{\pi}{3}, \frac{5\pi}{3}\right\}\bigg\backslash\big(\mathbb{D}_r(0) \cup \mathbb{D}_r(1)\big).$$

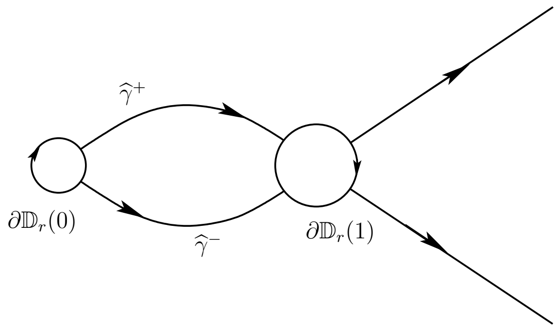

FIGURE 14. The oriented jump contours for the ratio function $\mathbf{R}(\lambda;t,\alpha,v)$ in the complex λ-plane.

(3) *As* $\lambda\to\infty$, *we have* $\mathbf{R}(\lambda)\to\mathbb{I}$.

Upon return to RHP 6.7 and RHP 6.9 we see that

$$\widehat{\mathbf{G}}_{\mathbf{R}}(\lambda;t,\alpha,v)=t^{\frac{1}{4}\sigma_3}\mathbf{G}_{\mathbf{R}}(\lambda;t,\alpha,v)t^{-\frac{1}{4}\sigma_3},\quad \lambda\in\Sigma_{\mathbf{R}}$$

satisfies the following small norm estimate

Proposition 6.12. *For any fixed* $\alpha>-1, v\in[0,+\infty)$ *there exist* $t_0=t_0(\alpha,v)>0$ *and* $c=c(v)>0$ *such that*

$$\|\widehat{\mathbf{G}}_{\mathbf{R}}(\cdot;t,\alpha,v)-\mathbb{I}\|_{L^2\cap L^\infty(\partial\mathbb{D}_r(0)\cup\partial\mathbb{D}_r(1))}\leq\frac{c}{\sqrt{t}},\qquad\forall\, t\geq t_0.$$

Moreover, recalling (6.11) and (6.12) together with (6.13) we see that $\widehat{\mathbf{G}}_{\mathbf{R}}(\lambda;t,\alpha,v)$ is exponentially close to the identity matrix (as $t\to+\infty$ and $v\in[0,+\infty),\alpha\in(-1,+\infty)$ are fixed) on $\widehat{\gamma}^{\pm}$ and the two contours extending to infinity, see Figure 14. Thus all together, cf. [26], we have

Theorem 6.13. *Given* $\alpha>-1, v\in[0,+\infty)$ *there exist* $t_0=t_0(\alpha,v)$ *and* $c=c(\alpha,v)$ *positive such that RHP 6.11 is uniquely solvable in* $L^2(\Sigma_{\mathbf{R}})$ *for all* $t\geq t_0$. *Its solution* $\mathbf{R}(\lambda)=\mathbf{R}(\lambda;t,\alpha,v)$ *can be computed iteratively via the integral equation*

$$\widehat{\mathbf{R}}(\lambda)=\mathbb{I}+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_{\mathbf{R}}}\widehat{\mathbf{R}}_-(w)\big(\widehat{\mathbf{G}}_{\mathbf{R}}(w)-\mathbb{I}\big)\frac{\mathrm{d}w}{w-\lambda},\quad\lambda\in\mathbb{C}\backslash\Sigma_{\mathbf{R}};\qquad \widehat{\mathbf{R}}(\lambda;t,\alpha,v)=t^{\frac{1}{4}\sigma_3}\mathbf{R}(\lambda;t,\alpha,v)t^{-\frac{1}{4}\sigma_3}$$

using the estimate

$$\|\widehat{\mathbf{R}}_-(\cdot;t,\alpha,v)-\mathbb{I}\|_{L^2(\Sigma_{\mathbf{R}})}\leq\frac{c}{\sqrt{t}},\qquad\forall\, t\geq t_0.$$

At this point we return to (6.2) and (6.3).

6.2. **Extraction of asymptotics and proof of expansion** (1.35)**.** We split this subsection into several parts.

6.2.1. *Preliminary expansions.* Recall the explicit and invertible transformation sequence

$$\mathbf{Y}(\lambda)\mapsto\mathbf{X}(\lambda)\mapsto\mathbf{T}(\lambda)\mapsto\mathbf{S}(\lambda)\mapsto\mathbf{R}(\lambda)$$

which leads us to the exact identity

$$\mathbf{Y}_1=\begin{bmatrix}1&0\\-b_1(\alpha)&1\end{bmatrix}\left\{\begin{bmatrix}2\nu^2&-\frac{2\mathrm{i}\nu}{\sqrt{t}}\\\frac{2\mathrm{i}}{3}\nu(1-4\nu^2)\sqrt{t}&-2\nu^2\end{bmatrix}+\frac{1}{t}\begin{bmatrix}a_2(\alpha)&-a_1(\alpha)\\b_1(\alpha)a_2(\alpha)-b_3(\alpha)&-a_2(\alpha)\end{bmatrix}\right.$$
$$\left.+\frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\mathbf{R}_-(w)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\,\mathrm{d}w\right\}\begin{bmatrix}1&0\\b_1(\alpha)&1\end{bmatrix},\tag{6.21}$$

where the coefficients $a_k(\alpha), b_k(\alpha)$ are defined in RHP B.1 below. Define

$$\mathbf{N} = \frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\,\mathrm{d}w, \qquad \mathbf{N} = \big(N^{jk}\big)_{j,k=1}^{2}$$

so that from a residue computation (using RHP 6.7 and 6.9)

Proposition 6.14. *As $t\to+\infty$,*

$$\begin{aligned} N^{11} &= \frac{2\mathrm{i}\nu}{\sqrt{t}}a_1(\alpha) - \frac{\mathrm{i}\nu}{\sqrt{t}}\sin\big(\eta(t,\alpha,v)\big) + \frac{2\nu^2}{\sqrt{t}}\big(\cos\big(\eta(t,\alpha,v)\big)-\mathrm{i}\nu\big) + \mathcal{O}\big(t^{-1}\big)\\ N^{12} &= \frac{1}{t}a_1(\alpha) - \frac{\nu^2}{t} - \frac{\mathrm{i}\nu}{t}\cos\big(\eta(t,\alpha,v)\big) + \mathcal{O}\big(t^{-\frac{3}{2}}\big). \end{aligned}$$

All error terms are uniform with respect to (α,v) chosen from compact subsets of $(-1,+\infty)\times[0,+\infty)$ and

$$\eta(t,\alpha,v) = 2\sqrt{t} - \frac{v}{2\pi}\ln(16t) - \pi\alpha + 2\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right).$$

Next we also require that from Theorem 6.13, RHP 6.7 and 6.9, as $t\to+\infty$,

$$\int_{\Sigma_{\mathbf{R}}}\big(\widehat{\mathbf{R}}_-(w)-\mathbb{I}\big)\big(\widehat{\mathbf{G}}_{\mathbf{R}}(w)-\mathbb{I}\big)\,\mathrm{d}w = \mathcal{O}\big(t^{-1}\big),$$

i.e. for

$$\mathbf{Q} = \frac{\mathrm{i}}{2\pi}\int_{\Sigma_{\mathbf{R}}}\big(\mathbf{R}_-(w)-\mathbb{I}\big)\big(\mathbf{G}_{\mathbf{R}}(w)-\mathbb{I}\big)\,\mathrm{d}w, \qquad \mathbf{Q} = \big(Q^{jk}\big)_{j,k=1}^{2}$$

we find in turn

Proposition 6.15. *As $t\to+\infty$ with fixed $(\alpha,v)\in(-1,+\infty)\times[0,+\infty)$,*

$$Q^{11} = \mathcal{O}\big(t^{-1}\big), \quad Q^{12} = \mathcal{O}\big(t^{-\frac{3}{2}}\big).$$

At this point we combine (6.21) and Propositions 6.14, 6.15,

Corollary 6.16. *As $t\to+\infty$,*

$$\begin{aligned} Y_1^{11} &= 2\nu^2 - \frac{\mathrm{i}\nu}{\sqrt{t}}\big(1+\sin\big(\eta(t,\alpha,v)\big)\big) + \frac{2\nu^2}{\sqrt{t}}\big(\cos\big(\eta(t,\alpha,v)\big)-\mathrm{i}\nu\big) + \mathcal{O}\left(t^{-1}\right)\\ Y_1^{12} &= -\frac{2\mathrm{i}\nu}{\sqrt{t}} - \frac{\nu^2}{t} - \frac{\mathrm{i}\nu}{t}\cos\big(\eta(t,\alpha,v)\big) + \mathcal{O}\big(t^{-\frac{3}{2}}\big); \end{aligned}$$

and all error terms are uniform with respect to (α,v) chosen from compact subsets of $(-1,+\infty)\times[0,+\infty)$.

Now back in (6.2) and (6.3),

Corollary 6.17. *As $t\to+\infty$ with fixed $(v,\alpha)\in[0,+\infty)\times(-1,+\infty)$,*

$$q^2(t,\alpha;\gamma) = \frac{2\mathrm{i}\nu}{\sqrt{t}}\Big(1-\sin\big(\eta(t,\alpha,v)\big)\Big) + \mathcal{O}\left(t^{-1}\right), \quad p^2(t,\alpha;\gamma) = 2\mathrm{i}\nu\sqrt{t}\Big(1+\sin\big(\eta(t,\alpha,v)\big)\Big) + \mathcal{O}(1)$$

and

$$\mathcal{H}_H\big(q(t,\alpha,\gamma),p(t,\alpha,\gamma),t,\alpha\big) = -\frac{v}{2\pi}\frac{1}{\sqrt{t}} + \frac{v^2}{8\pi^2}\frac{1}{t} - \frac{v}{4\pi t}\cos\big(\eta(t,\alpha,v)\big) + \mathcal{O}\big(t^{-\frac{3}{2}}\big)$$

The last result allows us already to determine all t-dependent terms in Theorem 1.7, expansion (1.35), indeed through Proposition A.1 we find

Corollary 6.18. *As $t\to+\infty$,*

$$\ln F_H(t,\alpha;\gamma) = -\frac{v}{\pi}\sqrt{t} + \frac{v^2}{8\pi^2}\ln t + D(\alpha,v) + \mathcal{O}\big(t^{-\frac{1}{2}}\big),$$

where $D(v,\alpha)$ is t-independent and the error term is uniform with respect to (α,v) chosen from compact subsets of $(-1,+\infty)\times[0,+\infty)$.

In order to determine $D(\alpha, v)$ we use (1.28) and first derive the following exact identity (recall RHP 6.2, (6.8), (6.9), (6.15), (6.20) and (B.2)), for $\lambda \in \mathbb{D}_r(0)$,

$$\widehat{\mathbf{X}}(\lambda) = \mathbf{R}(\lambda)\mathbf{E}^{(0)}(\lambda)\begin{bmatrix}1 & 0\\ b_1(\alpha) & 1\end{bmatrix}\Psi_\alpha(\lambda t)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\begin{bmatrix}1 & -\frac{\mathrm{i}}{2}\frac{1}{\sin\pi\alpha}\\ 0 & 1\end{bmatrix}(-\lambda)^{-\frac{\alpha}{2}\sigma_3}\mathrm{e}^{\frac{v}{2}\sigma_3},\quad \alpha\notin\mathbb{Z};$$

$$\widehat{\mathbf{X}}(\lambda) = \mathbf{R}(\lambda)\mathbf{E}^{(0)}(\lambda)\begin{bmatrix}1 & 0\\ b_1(\alpha) & 1\end{bmatrix}\Psi_\alpha(\lambda t)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\begin{bmatrix}1 & \frac{\mathrm{e}^{\mathrm{i}\pi\alpha}}{2\pi\mathrm{i}}\ln(-\lambda)\\ 0 & 1\end{bmatrix}(-\lambda)^{-\frac{\alpha}{2}\sigma_3}\mathrm{e}^{\frac{v}{2}\sigma_3},\quad \alpha\in\mathbb{Z}.$$

However, keeping in mind Remark B.2 below, we have for $\alpha > -1$,

$$\widehat{\mathbf{X}}(0) = \mathbf{R}(0)\mathbf{E}^{(0)}(0)\begin{bmatrix}1 & 0\\ b_1(\alpha) & 1\end{bmatrix}\widehat{\Psi}(0;\alpha)t^{\frac{\alpha}{2}\sigma_3}\mathrm{e}^{\frac{v}{2}\sigma_3},\quad \alpha\neq 0; \tag{6.22}$$

$$\widehat{\mathbf{X}}(0) = \mathbf{R}(0)\mathbf{E}^{(0)}(0)\begin{bmatrix}1 & 0\\ b_1(0) & 1\end{bmatrix}\widehat{\Psi}(0;0)\begin{bmatrix}1 & -\frac{\ln t}{2\pi\mathrm{i}}\\ 0 & 1\end{bmatrix}\mathrm{e}^{\frac{v}{2}\sigma_3},\qquad \alpha = 0. \tag{6.23}$$

But with Theorem 6.13,

$$\widehat{\mathbf{R}}(0) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_{\Sigma_{\mathbf{R}}}\big(\widehat{\mathbf{G}}_{\mathbf{R}}(w) - \mathbb{I}\big)\frac{\mathrm{d}w}{w} + \mathcal{O}\big(t^{-1}\big),$$

and by residue computation,

Proposition 6.19. *As $t \to +\infty$, for any fixed $\alpha > -1$ and $v \in [0, +\infty)$,*

$$\widehat{R}^{11}(0) = 1 + \frac{2\mathrm{i}\nu}{\sqrt{t}}a_1(1-4\nu^2) + \frac{\mathrm{i}\nu}{\sqrt{t}}\big(\sin\eta(t,\alpha,v) + 2\mathrm{i}\nu(\cos\eta(t,\alpha,v) - \mathrm{i}\nu)\big) + \mathcal{O}\big(t^{-1}\big),$$

$$\widehat{R}^{12}(0) = \frac{\nu^2}{\sqrt{t}}(1-4a_1) + \frac{\mathrm{i}\nu}{\sqrt{t}}\cos\big(\eta(t,\alpha,v)\big) + \mathcal{O}\big(t^{-1}\big)$$

Thus back in (6.22) and (6.23)

Corollary 6.20. *As $t \to +\infty$ with fixed $\alpha > -1, v \in [0, +\infty)$,*

$$\widehat{X}^{11}(0) = \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left(1 - \frac{\mathrm{i}\nu}{\sqrt{t}}\Big(1 - \sin\big(\eta(t,\alpha,v)\big)\Big) - \frac{2\mathrm{i}\nu}{\sqrt{t}}\alpha + \mathcal{O}\big(t^{-1}\big)\right)\frac{2^{-\alpha}t^{\frac{\alpha}{2}}\mathrm{e}^{\frac{v}{2}}}{\Gamma(1+\alpha)},$$

and for fixed $\alpha > -1, \alpha \neq 0, v \in [0, +\infty)$,

$$\widehat{X}^{12}(0) = \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left(1 - \frac{\mathrm{i}\nu}{\sqrt{t}}\Big(1 - \sin\big(\eta(t,\alpha,v)\big)\Big) + \frac{2\mathrm{i}\nu}{\sqrt{t}}\alpha + \mathcal{O}\big(t^{-1}\big)\right)\frac{\Gamma(\alpha)}{2\pi\mathrm{i}}2^{\alpha}t^{-\frac{\alpha}{2}}\mathrm{e}^{-\frac{v}{2}}.$$

The last Corollary allows us to compute parts of $L(t,\alpha;\gamma)$, see (6.4) and (6.5). For the remaining parts we evidently require the small t behavior of $\widehat{\mathbf{X}}(0)$.

6.2.2. *Small t-behavior of $\mathbf{Y}(\lambda; t, \alpha, \gamma)$ for $\alpha > 0$.* We return to RHP 6.1 and use the power series expansion

$$J_\alpha(z) = \left(\frac{z}{2}\right)^{\alpha}\left\{\frac{1}{\Gamma(1+\alpha)} + \mathcal{O}\big(z^2\big)\right\},\quad z\to 0,\quad z\notin(-\infty,0] \tag{6.24}$$

to obtain the following small norm estimate.

Proposition 6.21. *For any fixed $\alpha > 0$ there exist $t_0 = t_0(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$\|\mathbf{G}_{\mathbf{Y}}(\cdot; t, \alpha, v) - \mathbb{I}\|_{L^2\cap L^\infty(0,1)} \leq c\,t^{\alpha},\qquad \forall\, t \leq t_0,\qquad \gamma\in[0,1].$$

Hence, by general theory [26], the initial RHP 6.1 is solvable as $t \downarrow 0$ and $\alpha > 0$

Theorem 6.22. *Given $\alpha > 0$ there exist $t_0 = t_0(\alpha)$ and $c = c(\alpha)$ positive such that RHP 6.1 is solvable in $L^2(0,1)$ for all $t \leq t_0$. Its solution $\mathbf{Y}(\lambda) = \mathbf{Y}(\lambda; t, \alpha, v)$ can be computed iteratively via the integral equation*

$$\mathbf{Y}(\lambda) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_0^1 \mathbf{Y}_-(w)\big(\mathbf{G}_{\mathbf{Y}}(w) - \mathbb{I}\big)\frac{\mathrm{d}w}{w-\lambda},\quad \lambda\in\mathbb{C}\backslash[0,1]$$

using the estimate

$$\|\mathbf{Y}_-(\cdot; t, \alpha, v) - \mathbb{I}\|_{L^2(0,1)} \leq c\,t^{\alpha},\quad \forall\ t \leq t_0,\quad \gamma\in[0,1].$$

Combining this last result with RHPs 6.2 and B.1 we find in turn

Corollary 6.23. *As $t \downarrow 0$ with fixed $\alpha > 0$,*

$$\widehat{X}^{11}(0) = \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left(1+\mathcal{O}\left(t^{\min\{\alpha,1\}}\right)\right)\frac{2^{-\alpha}t^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)},\quad \widehat{X}^{12}(0) = \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left(1+\mathcal{O}\left(t^{\min\{\alpha,1\}}\right)\right)\frac{\Gamma(\alpha)}{2\pi\mathrm{i}}\,2^{\alpha}t^{-\frac{\alpha}{2}},$$

uniformly for any $\gamma \in [0,1]$.

6.2.3. *Small t-behavior for $-1 < \alpha < 0$.* For this parameter regime we again return to RHP 6.1 but apply first the following transformation

$$\mathbf{W}(\lambda) = t^{-\frac{\alpha}{2}\sigma_3}\big(\widehat{\Psi}(0;\alpha)\big)^{-1}\mathbf{Y}(\lambda)\begin{cases}\widehat{\Psi}(\lambda t;\alpha)t^{\frac{\alpha}{2}\sigma_3}\mathbf{M}(\lambda), & \lambda \in \mathbb{D}_r(\frac{1}{2})\\ \widehat{\Psi}(0;\alpha)t^{\frac{\alpha}{2}\sigma_3}\mathbf{M}(\lambda), & \lambda \notin \mathbb{D}_r(\frac{1}{2})\end{cases},\qquad \frac{1}{2} < r < 1\ \text{ fixed.}$$

The entire function $\widehat{\Psi}(\zeta;\alpha)$ is defined in RHP B.1 below and we have introduced

$$\mathbf{M}(\lambda) = \begin{bmatrix}1 & \frac{\gamma}{2\pi\mathrm{i}}\int_0^1\frac{w^{\alpha}}{w-\lambda}\,\mathrm{d}w\\ 0 & 1\end{bmatrix},\qquad \lambda \in \mathbb{C}\backslash[0,1],\qquad \gamma\in[0,1],\ \ -1<\alpha<0. \tag{6.25}$$

Riemann-Hilbert Problem 6.24. *The function $\mathbf{M}(\lambda) = \mathbf{M}(\lambda;\alpha,\gamma) \in \mathbb{C}^{2\times 2}$ defined in (6.25) has the following properties*

(1) *$\mathbf{M}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[0,1]$.*
(2) *Orienting the interval $[0,1] \subset \mathbb{R}$ from left to right we have*

$$\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda)\begin{bmatrix}1 & \gamma\lambda^{\alpha}\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & \gamma\lambda^{\alpha}\\ 0 & 1\end{bmatrix}\mathbf{M}_-(\lambda),\qquad \lambda \in (0,1).$$

(3) *$\mathbf{M}(\lambda)$ is square integrable on $[0,1] \subset \mathbb{R}$, in more detail for $\alpha \in (-1,0)$,*

$$\mathbf{M}(\lambda) = \widehat{\mathbf{M}}(\lambda)\begin{bmatrix}1 & \frac{\mathrm{i}\gamma}{2}\frac{(-\lambda)^{\alpha}}{\sin\pi\alpha}\\ 0 & 1\end{bmatrix},\qquad \lambda\to 0,\ \lambda\notin[0,+\infty),$$

where z^{α} is defined with its principal branch. Here, $\widehat{\mathbf{M}}(\lambda)$ is analytic at $\lambda = 0$,

$$\widehat{\mathbf{M}}(\lambda) = \mathbb{I} + \frac{\gamma}{2\pi\mathrm{i}\alpha}\sigma_+ + \mathcal{O}\big(\lambda\big),\qquad \lambda\to 0.$$

(4) *As $\lambda\to\infty$,*

$$\mathbf{M}(\lambda) = \mathbb{I} + \frac{\mathrm{i}\gamma}{2\pi}\frac{\sigma_+}{\alpha+1}\frac{1}{\lambda} + \mathcal{O}\left(\lambda^{-2}\right).$$

At this point we recall RHP 6.1 and make use of (6.6) in order to derive the following RHP for $\mathbf{Z}(\lambda)$

Riemann-Hilbert Problem 6.25. *Find $\mathbf{W}(\lambda) = \mathbf{W}(\lambda;t,\alpha,v) \in \mathbb{C}^{2\times 2}$ such that*

(1) *$\mathbf{W}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash\partial\mathbb{D}_r(\frac{1}{2})$ and we orient $\partial\mathbb{D}_r(\frac{1}{2})$ clockwise.*
(2) *By construction, compare (6.6), $\mathbf{W}(\lambda)$ has no jump on the interval $(0,1)\subset\mathbb{R}$. Instead we observe that*

$$\mathbf{W}_+(\lambda) = \mathbf{W}_-(\lambda)\mathbf{M}^{-1}(\lambda)t^{-\frac{\alpha}{2}\sigma_3}\big(\widehat{\Psi}(\lambda t;\alpha)\big)^{-1}\widehat{\Psi}(0;\alpha)t^{\frac{\alpha}{2}\sigma_3}\mathbf{M}(\lambda),\qquad \lambda\in\partial\mathbb{D}_r\left(\frac{1}{2}\right).$$

(3) *$\mathbf{W}(\lambda)$ is bounded at $\lambda = 0$ and $\lambda = 1$.*
(4) *As $\lambda\to\infty$ we have*

$$\mathbf{W}(\lambda) = \mathbb{I} + \mathcal{O}\big(\lambda^{-1}\big).$$

Since $\mathbf{M}(\lambda)$ is t-independent and $\widehat{\Psi}(\lambda t;\alpha)$ analytic at $\lambda = 0$, we obtain at once

Proposition 6.26. *For any fixed $\alpha\in(-1,0)$ there exist $t_0 = t_0(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$\|\mathbf{G}_{\mathbf{W}}(\cdot;t,\alpha,v) - \mathbb{I}\|_{L^2\cap L^{\infty}(\partial\mathbb{D}_r(\frac{1}{2}))} \leq c\,t^{1+\alpha},\qquad \forall\, t\leq t_0,\qquad \gamma\in[0,1].$$

In short, the transformed problem 6.25 is solvable as $t\downarrow 0$ and $\alpha\in(-1,1)$, cf. [26].

Theorem 6.27. *Given $\alpha \in (-1,0)$ there exist $t_0 = t_0(\alpha)$ and $c = c(\alpha)$ positive such that RHP 6.25 is solvable in $L^2(0,1)$ for all $t \leq t_0$. Its solution $\mathbf{W}(\lambda) = \mathbf{W}(\lambda; t, \alpha, v)$ can be computed iteratively via the integral equation*

$$\mathbf{W}(\lambda) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}} \oint_{\partial\mathbb{D}_r(\frac{1}{2})} \mathbf{W}_-(w)\big(\mathbf{G}_{\mathbf{W}}(w) - \mathbb{I}\big)\frac{\mathrm{d}w}{w-\lambda}, \quad \lambda \in \mathbb{C}\backslash\partial\mathbb{D}_r\left(\frac{1}{2}\right)$$

using the estimate

$$\|\mathbf{W}_-(\cdot; t, \alpha, v) - \mathbb{I}\|_{L^2(\partial\mathbb{D}_r(\frac{1}{2}))} \leq c\, t^{1+\alpha}, \qquad \forall\ t \leq t_0, \quad \gamma \in [0,1].$$

This last result allows us to derive small t-expansions for $\widehat{\mathbf{X}}(0)$, compare RHP 6.2, B.1 and 6.24.

Corollary 6.28. *As $t \downarrow 0$ with fixed $\alpha \in (-1,0)$,*

$$\begin{aligned}
\widehat{X}^{11}(0) &= \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left(1 + \mathcal{O}\left(t^{1+\alpha}\right)\right)\frac{2^{-\alpha}t^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)},\\
\widehat{X}^{12}(0) &= \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left(1 + \mathcal{O}\left(t^{\min\{1+\alpha,-\alpha\}}\right)\right)\left(-\frac{\gamma}{2\pi\mathrm{i}\alpha}\right)\frac{2^{-\alpha}t^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)}.
\end{aligned}$$

6.2.4. *Derivation of* (1.35). We know from Corollary 6.17 that as $t \to +\infty$ and $(v,\alpha) \in [0,+\infty) \times (-1,+\infty)$ are fixed,

$$2t\,\mathcal{H}_H(q,p,t,\alpha) = -\frac{v}{\pi}\sqrt{t} + \frac{v^2}{4\pi^2} - \frac{v}{2\pi}\cos\eta(t,\alpha,v) + \mathcal{O}\big(t^{-\frac{1}{2}}\big). \tag{6.26}$$

On the other hand, via the same Corollary 6.17 and through standard manipulations with trigonometric functions,

Corollary 6.29. *As $t \to +\infty$ with fixed $(v,\alpha) \in [0,+\infty) \times (-1,+\infty)$,*

$$pq_\gamma = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\gamma}\left\{v\cos\big(\eta(t,\alpha,v)\big) + \frac{v^2}{4\pi}\ln(16t)\right\} - \frac{v}{\pi}\frac{\mathrm{d}}{\mathrm{d}\gamma}\arg\Gamma\left(\frac{\mathrm{i}v}{2\pi}\right) + \mathcal{O}\left(t^{-\frac{1}{2}}\ln t\right).$$

Next, through Corollaries 6.20 and 6.23,

Corollary 6.30. *As $t \to +\infty$ with fixed $(v,\alpha) \in [0,+\infty) \times (-1,+\infty)$,*

$$\begin{aligned}
L(t,\alpha;\gamma) &= -\frac{\alpha}{2}v + \mathcal{O}\big(t^{-\frac{1}{2}}\big), \quad \alpha > 0;\\
L(t,\alpha;\gamma) &= -\frac{\alpha^2}{2}\ln t - \frac{\alpha}{2}v - \frac{\alpha}{2}\ln(-\gamma) + \alpha\ln\big(2^\alpha\Gamma(1+\alpha)\big) + \mathcal{O}\big(t^{-\frac{1}{2}}\big),\ -1 < \alpha < 0.
\end{aligned}$$

At this point we can start to determine I_H through Corollary 1.6,

Corollary 6.31. *As $t \to +\infty$ with fixed $(v,\alpha) \in [0,+\infty) \times (-1,+\infty)$,*

$$I_H(t,\alpha;\gamma) = \frac{v}{2\pi}\cos\eta(t,\alpha,v) + \frac{v^2}{8\pi^2}\ln(16t) - \frac{1}{\pi}\int_0^\gamma v(\gamma')\frac{\mathrm{d}}{\mathrm{d}\gamma'}\arg\Gamma\left(\frac{\mathrm{i}v(\gamma')}{2\pi}\right)\mathrm{d}\gamma' + G(t) + \mathcal{O}\big(t^{-\frac{1}{2}}\big),$$

for $\alpha \geq 0$, where $G(t)$ is (α,v)-independent. On the other hand, for $-1 < \alpha < 0$, as $t \to +\infty$,

$$\begin{aligned}
I_H(t,\alpha;\gamma) = \frac{v}{2\pi}\cos\eta(t,\alpha,v) &+ \frac{v^2}{8\pi^2}\ln(16t) - \frac{1}{\pi}\int_0^\gamma v(\gamma')\frac{\mathrm{d}}{\mathrm{d}\gamma'}\arg\Gamma\left(\frac{\mathrm{i}v(\gamma')}{2\pi}\right)\mathrm{d}\gamma' - \frac{\alpha}{2}\ln(-\gamma) - \frac{\alpha^2}{2}\ln t\\
&+ \alpha\ln\big(2^\alpha\Gamma(1+\alpha)\big) + H(t) + \mathcal{O}\big(t^{-\frac{1}{2}}\big),
\end{aligned}$$

where $H(t)$ is (α,v)-independent.

Combining the last two Corollaries with (6.26) and Corollary 6.18 (using also $F_H(t,\alpha;0) = 1$ again) we have thus back in (1.28),

Proposition 6.32. *As $t \to +\infty$,*

$$\ln F_H(t,\alpha;\gamma) = -\frac{v}{\pi}\sqrt{t} + \frac{v^2}{8\pi^2}\ln(16t) + \frac{\alpha}{2}v + \frac{v^2}{4\pi^2} - \frac{1}{\pi}\int_0^\gamma v(\gamma')\frac{\mathrm{d}}{\mathrm{d}\gamma'}\arg\Gamma\left(\frac{\mathrm{i}v(\gamma')}{2\pi}\right)\mathrm{d}\gamma' + \mathcal{O}\big(t^{-\frac{1}{2}}\big),$$

uniformly for fixed $(\alpha,v) \in (-1,+\infty) \times [0,+\infty)$.

This expansion is exactly equal to (1.35) once we recall again standard properties of the Barnes-G function.

Appendix A. Differential equations

The kernel (1.25) identifies the corresponding integral operator as integrable in the sense of [45] and a simple rescaling argument identifies RHP 6.1 subsequently as the underlying RHP. Moreover, following [45, 24] (see also [15]) we find at once the following differential identity

Proposition A.1. *For any fixed $\alpha > -1$ and $\gamma \in [0,1]$ we have*

$$\frac{\partial}{\partial t}\ln F_H(t,\alpha;\gamma) = \frac{1}{2}Y_1^{12}$$

in terms of the solution to RHP 6.1, see (6.1).

In order to characterize (q,p) in (1.13), (1.19) via RHP 6.1 we apply the following standard argument: First return to RHP 6.2 and note that all jumps in the same problem are λ- and t-independent. Hence the functions $\frac{\partial \mathbf{Z}}{\partial\lambda}\mathbf{Z}^{-1}$ and $\frac{\partial \mathbf{Z}}{\partial t}\mathbf{Z}^{-1}$ with

$$\mathbf{Z}(\lambda) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{8}(3+4\alpha^2) & 1 \end{bmatrix}\mathbf{X}(\lambda), \quad \lambda \in \mathbb{C}\backslash\Sigma_{\mathbf{X}}$$

are meromorphic in $\lambda \in \mathbb{C}\backslash\{0,1\}$. In fact, using (6.7) together with (6.1) we find that

$$\begin{aligned}\frac{\partial \mathbf{Z}}{\partial\lambda}\mathbf{Z}^{-1} = &-\frac{t}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} + \frac{1}{2\lambda}\begin{bmatrix} -tY_1^{12} & 1 \\ 2tY_1^{11}+\alpha^2 & tY_1^{12}\end{bmatrix} \\ &+\frac{1}{\lambda^2}\begin{bmatrix} -Y_1^{11}+\frac{1}{2}(Y_1^{12}\alpha^2-Y_1^{21})-\frac{t}{2}Y_2^{12}+\frac{t}{2}Y_1^{11}Y_1^{12} & -Y_1^{12}+Y_1^{11}+\frac{t}{2}(Y_1^{12})^2 \\ -Y_1^{21}-Y_1^{11}\alpha^2-\frac{t}{2}(Y_2^{22}-Y_2^{11})+\frac{t}{2}(1-(Y_1^{11})^2) & Y_1^{11}-\frac{1}{2}(Y_1^{12}\alpha^2-Y_1^{21})+\frac{t}{2}Y_2^{12}-\frac{t}{2}Y_1^{11}Y_1^{12}\end{bmatrix} \\ &+\mathcal{O}\left(\lambda^{-3}\right), \qquad \lambda\to\infty.\end{aligned} \tag{A.1}$$

Likewise from RHP 6.2, condition (3), we obtain as $\lambda \to 0$,

$$\frac{\partial \mathbf{Z}}{\partial\lambda}\mathbf{Z}^{-1}\bigg|_{\alpha\neq 0} = \frac{\alpha}{2\lambda}\widehat{\mathbf{Z}}(0)\sigma_3\big(\widehat{\mathbf{Z}}(0)\big)^{-1}+\mathcal{O}(1); \quad \frac{\partial \mathbf{Z}}{\partial\lambda}\mathbf{Z}^{-1}\bigg|_{\alpha=0} = -\frac{1-\gamma}{2\pi\mathrm{i}\lambda}\widehat{\mathbf{Z}}(0)\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\big(\widehat{\mathbf{Z}}(0)\big)^{-1}+\mathcal{O}(1) \tag{A.2}$$

where $\widehat{\mathbf{Z}}(\lambda) = \left[\begin{smallmatrix} 1 & 0 \\ -\frac{1}{8}(3+4\alpha^2) & 1\end{smallmatrix}\right]\widehat{\mathbf{X}}(\lambda)$. Finally, from condition (4) in RHP 6.2,

$$\frac{\partial \mathbf{Z}}{\partial\lambda}\mathbf{Z}^{-1} = \frac{\gamma}{2\pi\mathrm{i}}\widehat{\mathbf{Z}}(1)\begin{bmatrix} -1 & -\mathrm{e}^{-\mathrm{i}\pi\alpha} \\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{bmatrix}\big(\widehat{\mathbf{Z}}(1)\big)^{-1}\frac{1}{\lambda-1}+\mathcal{O}(1), \qquad \lambda\to 1. \tag{A.3}$$

Combining (A.1),(A.2) and (A.3) thus

$$\frac{\partial \mathbf{Z}}{\partial\lambda} = \left\{-\frac{t}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}+\frac{\mathbf{A}}{\lambda-1}+\frac{\mathbf{B}}{\lambda}\right\}\mathbf{Z}, \tag{A.4}$$

where we parametrize the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} = \begin{bmatrix} uv & -v^2 \\ u^2 & -uv\end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix}\gamma_1 & \gamma_3 \\ \gamma_2 & -\gamma_1\end{bmatrix}: \ \ \gamma_1^2+\gamma_2\gamma_3 = \frac{\alpha^2}{4}, \tag{A.5}$$

with $u = u(t,\alpha,\gamma), v = v(t,\alpha,\gamma), \gamma_j = \gamma_j(t,\alpha,\gamma) \in \mathbb{C}$. Note that we have with (A.1),

$$\mathbf{A}+\mathbf{B} = \frac{1}{2}\begin{bmatrix} -tY_1^{12} & 1 \\ 2tY_1^{11}+\alpha^2 & tY_1^{12}\end{bmatrix}, \tag{A.6}$$

and

$$\mathbf{A} = \begin{bmatrix} -Y_1^{11}+\frac{1}{2}(Y_1^{12}\alpha^2-Y_1^{21})-\frac{t}{2}Y_2^{12}+\frac{t}{2}Y_1^{11}Y_1^{12} & -Y_1^{12}+Y_1^{11}+\frac{t}{2}(Y_1^{12})^2 \\ -Y_1^{21}-Y_1^{11}\alpha^2-\frac{t}{2}(Y_2^{22}-Y_2^{11})+\frac{t}{2}(1-(Y_1^{11})^2) & Y_1^{11}-\frac{1}{2}(Y_1^{12}\alpha^2-Y_1^{21})+\frac{t}{2}Y_2^{12}-\frac{t}{2}Y_1^{11}Y_1^{12}\end{bmatrix}. \tag{A.7}$$

This allows us to compute all coefficients in (A.4) through (6.1). Next, using again (6.7) and (6.1) we also find

$$\frac{\partial \mathbf{Z}}{\partial t}\mathbf{Z}^{-1} = -\frac{\lambda}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}+\frac{1}{2t}\begin{bmatrix} -tY_1^{12} & 1 \\ 2tY_1^{11}+\alpha^2 & tY_1^{12}\end{bmatrix}+\frac{1}{\lambda}\left(\frac{1}{t}(\mathbf{A}+\mathbf{Y}_1)+(\mathbf{Y}_1)_t\right)+\mathcal{O}\left(\lambda^{-2}\right), \quad \lambda\to\infty$$

and together with $\frac{\partial \mathbf{Z}}{\partial t}\mathbf{Z}^{-1}=\mathcal{O}(1)$ as $\lambda\to 0$ and $\lambda\to 1$ we have in addition to (A.4) also

$$\frac{\partial \mathbf{Z}}{\partial t}=\left\{-\frac{\lambda}{2}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+\frac{1}{t}(\mathbf{A}+\mathbf{B})\right\}\mathbf{Z}\quad \text{and}\quad \frac{1}{t}(\mathbf{A}+\mathbf{Y}_1)+(\mathbf{Y}_1)_t\equiv 0. \tag{A.8}$$

Frobenius integrability of the overdetermined system (A.4), (A.8) leads to the zero curvature condition which in turn is equivalent to

$$\frac{\partial \mathbf{A}}{\partial t}=\frac{1}{2}\Big[\mathbf{A},\left[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right]\Big]-\frac{1}{t}[\mathbf{A},\mathbf{B}],\qquad \frac{\partial \mathbf{B}}{\partial t}=\frac{1}{t}[\mathbf{A},\mathbf{B}]. \tag{A.9}$$

Translating (A.9) into the corresponding matrix entries we find a coupled nonlinear system for $(u,v,\gamma_1,\gamma_2,\gamma_3)$,

$$(\gamma_1)_t=-\frac{1}{t}(v^2\gamma_2+u^2\gamma_3),\quad (\gamma_2)_t=\frac{2}{t}(u^2\gamma_1-uv\gamma_2),\quad (\gamma_3)_t=\frac{2}{t}(v^2\gamma_1+uv\gamma_3), \tag{A.10}$$

$$(uv)_t=-\frac{v^2}{2}+\frac{1}{t}(v^2\gamma_2+u^2\gamma_3),\quad (v^2)_t=\frac{2}{t}(v^2\gamma_1+uv\gamma_3),\quad (u^2)_t=-uv-\frac{2}{t}(u^2\gamma_1-uv\gamma_2). \tag{A.11}$$

Recall that from (A.5) and (A.6)

$$uv+\gamma_1=-\frac{t}{2}Y_1^{12},\qquad u^2+\gamma_2=tY_1^{11}+\frac{\alpha^2}{2},\qquad -v^2+\gamma_3=\frac{1}{2}. \tag{A.12}$$

Remark A.2. *In view of* (A.2) *and* (A.5) *as well as* (A.6) *we can parametrize the diagonalizing matrix* $\widehat{\mathbf{Z}}(0)$ *as*

$$\widehat{\mathbf{Z}}(0)=\alpha^{-\frac{1}{2}}\begin{bmatrix}1 & -\left(\frac{1}{2}+v^2\right)\\ (\frac{t}{2}Y_1^{12}+uv+\frac{\alpha}{2})(\frac{1}{2}+v^2)^{-1} & \frac{\alpha}{2}-\frac{t}{2}Y_1^{12}-uv\end{bmatrix}\left(\frac{1}{2}+v^2\right)^{\frac{1}{2}\sigma_3}w^{\sigma_3},\qquad \alpha\neq 0, \tag{A.13}$$

with some $w=w(t,\alpha,\gamma)\in\mathbb{C}\backslash\{0\}$. *But* (A.8) *tells us that*

$$\frac{\partial \widehat{\mathbf{Z}}}{\partial t}(0)=\frac{1}{t}(\mathbf{A}+\mathbf{B})\widehat{\mathbf{Z}}(0), \tag{A.14}$$

hence substituting (A.13) *into* (A.14) *we find with the help of* (A.10) *and* (A.11),

$$\frac{\partial}{\partial t}\ln w=\frac{\alpha}{4t}\frac{1}{\frac{1}{2}+v^2},\qquad\qquad w^2=-\widehat{Z}^{11}(0)\big(\widehat{Z}^{12}(0)\big)^{-1}=-\widehat{X}^{11}(0)\big(\widehat{X}^{12}(0)\big)^{-1}.$$

In addition we also have

$$\frac{\partial}{\partial t}\ln\left(w\,t^{-\frac{1}{2}\alpha}\right)=-\frac{\alpha}{2t}\frac{v^2}{\frac{1}{2}+v^2}.$$

The above identities allow us to replace γ_j in (A.10): first, using the formula for γ_1 and γ_3 in the third equation of (A.10) we find

$$v\left(tv_t+\frac{t}{2}vY_1^{12}-\frac{u}{2}\right)\equiv 0\quad\forall t.$$

But if v were to vanish identically, then $\gamma_3\equiv\frac{1}{2}$ and $u\equiv 0$, see (A.11). Hence all γ_j are t-independent and we find from (A.5) that $Y_1^{11}+\frac{t}{2}(Y_1^{12})^2\equiv 0$. But now $\mathbf{A}\equiv\mathbf{0}$ so that with (A.7), $Y_1^{12}\equiv 0$, which contradicts Proposition A.1. In short, we have the differential equation

$$tv_t=-\frac{t}{2}vY_1^{12}+\frac{u}{2}. \tag{A.15}$$

Second, using the formulæ for γ_1,γ_2 and γ_3 in the first equation of (A.10) we find

$$\left(uv+\frac{t}{2}Y_1^{12}\right)_t=\frac{1}{t}\left(v^2tY_1^{11}+v^2\frac{\alpha^2}{2}+\frac{u^2}{2}\right).$$

But $(tY_1^{12})_t=v^2$ (adding the first two equations in (A.10), (A.11) and using (A.6)) so together with (A.15) we find from the last equation that

$$tu_t=\frac{1}{2}(\alpha^2-t)v+vtY_1^{11}+\frac{1}{2}utY_1^{12}. \tag{A.16}$$

Incidentally, using the formulæ for γ_1 and γ_2 in the second equation of (A.10) one also obtains (A.16) upon recalling that $(tY_1^{11})_t=-uv$ (adding the last two equations in (A.10), (A.11) and using (A.6)). We summarize

Proposition A.3 (see [34], Proposition 9.5.2)**.** *The functions* $(u, v, Y_1^{11}, Y_1^{12})$ *defined in* (6.1) *and* (A.5), (A.7) *satisfy the nonlinear dynamical system*

$$tv_t = -\frac{t}{2}vY_1^{12} + \frac{u}{2}, \quad tu_t = \frac{1}{2}(\alpha^2 - t)v + vtY_1^{11} + \frac{1}{2}utY_1^{12}, \quad (tY_1^{12})_t = v^2, \qquad (tY_1^{11})_t = -uv. \tag{A.17}$$

Moreover, the combination $\sigma = \sigma(t) = \frac{1}{2}tY_1^{12}$ *solves the Jimbo-Miwa-Okamoto form of Painlevé III,*

$$(t\sigma_{tt})^2 - \alpha^2(\sigma_t)^2 - \sigma_t(1 + 4\sigma_t)(\sigma - t\sigma_t) = 0. \tag{A.18}$$

Proof. The derivation of the differential equation (A.18) for σ is nearly identical to [34], Chapter 9.5, see also [65]. Apply $t\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of the first equation in (A.17) and use the second, third and fourth in the resulting right hand side,

$$t(tv_t)_t = \frac{1}{4}(\alpha^2 - t)v + \frac{v}{2}\left(tY_1^{11} + \frac{1}{2}(tY_1^{12})^2\right) - \frac{t}{2}v^3. \tag{A.19}$$

But from (A.8) we have $(t\mathbf{Y}_1)_t = -\mathbf{A}$ which evaluated at its 12 entry, see (A.7), gives

$$v^2 = Y_1^{12} - \left(Y_1^{11} + \frac{1}{2}t(Y_1^{12})^2\right).$$

Multiplying the last identity by t we then rewrite (A.19) as

$$t(tv_t)_t = \frac{1}{4}(\alpha^2 - t)v + \frac{v}{2}tY_1^{12} - tv^3. \tag{A.20}$$

Next, we multiply (A.20) by $2v_t$ and use the third equation in (A.17),

$$\frac{\mathrm{d}}{\mathrm{d}t}(tv_t)^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}(\alpha^2 - t + 2tY_1^{12})v^2 - \frac{t}{2}v^4 + \frac{t}{4}Y_1^{12}\right),$$

so that

$$-\frac{1}{2}tY_1^{12}\left(v^2 + \frac{1}{2}\right) = \frac{1}{4}(\alpha^2 - t)v^2 - \frac{t}{2}v^4 - (tv_t)^2 + C \tag{A.21}$$

involving a t-independent expression C. Combining (A.21) with the third equation in (A.17) we finally obtain

$$(t\sigma_{tt})^2 - \alpha^2(\sigma_t)^2 - \sigma_t(1 + 4\sigma_t)(\sigma - t\sigma_t) = 2C\sigma_t. \tag{A.22}$$

We will now argue that in fact $C = 0$: From the Fredholm series of the Bessel kernel determinant we compute directly the following boundary behavior (which is differentiable with respect to t),

$$\det(1 - \gamma K_{\mathrm{Bess}}^{\alpha}\restriction_{L^2(0,t)}) = 1 - \frac{\gamma\, t^{\alpha+1}}{2^{2\alpha+2}\Gamma^2(2+\alpha)} + \mathcal{O}(t^{\alpha+2}), \quad t \downarrow 0.$$

Hence, by Proposition A.1 and the definition of $\sigma(t)$,

$$\sigma(t) = -\frac{\gamma\, t^{\alpha+1}}{2^{2\alpha+2}\Gamma(1+\alpha)\Gamma(2+\alpha)} + \mathcal{O}(t^{\alpha+2}), \qquad t \downarrow 0.$$

But once we substitute this expansion into (A.22) a balance of exponents can only be achieved for $C = 0$. ☐

Observe that upon multiplication of (A.20) with $v^2 + \frac{1}{2}$ and the use of (A.21) for the term $tY_1^{12}(v^2 + \frac{1}{2})$ we arrive at

$$t\left(v^2 + \frac{1}{2}\right)(tv_t)_t = v(tv_t)^2 + \frac{1}{4}(\alpha^2 - t)\frac{v}{2} - \frac{t}{2}v^3(v^2 + 1).$$

Provided we let $q = q(t) = \pm\mathrm{i}\sqrt{2}\,v(t)$ (with either choice of the sign), then the last differential equation is equivalent to

$$t(q^2 - 1)(tq_t)_t = q(tq_t)^2 + \frac{1}{4}(t - \alpha^2)q + \frac{1}{4}tq^3(q^2 - 2) \tag{A.23}$$

which is exactly (1.16) in [65]. In addition, if we put $\mathcal{H}_H = \frac{1}{2}Y_1^{12}$ then (A.21) shows that (A.23) can be reformulated as a Hamiltonian dynamical system

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial\mathcal{H}_H}{\partial q}, \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial\mathcal{H}_H}{\partial p}, \qquad \mathcal{H}_H = \mathcal{H}_H(q, p, t) = \frac{q^2 - 1}{4t}p^2 - \frac{(\alpha^2 - t)q^2 + tq^4}{4t(q^2 - 1)}$$

where (see (A.5), (A.7))

$$q^2 = t(Y_1^{12})^2 + 2(Y_1^{11} - Y_1^{12}), \qquad p^2 = \frac{\alpha^2 q^2}{(q^2-1)^2} + \frac{2t}{q^2-1}\left(Y_1^{12} + \frac{q^2}{2}\right),$$

or equivalently (see (A.12) and (A.2), (A.5))

$$q^2 = 1 - 2\gamma_3, \qquad \gamma_3 = -\alpha \widehat{X}^{11}(0)\widehat{X}^{12}(0),\ \alpha \neq 0; \qquad \gamma_3 = -\frac{1-\gamma}{2\pi \mathrm{i}}\big(\widehat{X}^{11}(0)\big)^2,\ \alpha = 0.$$

Moreover, recalling Remark A.2, we also have that

$$\frac{\partial}{\partial t}\ln w = -\frac{\alpha}{2t}\frac{1}{q^2-1}, \qquad \frac{\partial}{\partial t}\ln\left(wt^{-\frac{1}{2}\alpha}\right) = -\frac{\alpha}{2t}\frac{q^2}{q^2-1}, \quad w^2 = -\widehat{X}^{11}(0)\big(\widehat{X}^{12}(0)\big)^{-1},$$

and thus

$$L(t,\alpha;\gamma) = -\frac{\alpha}{2}\ln\Big(-\widehat{X}^{11}(0)\big(\widehat{X}^{12}(0)\big)^{-1}\Big)\Big|_{s=0}^{t}, \qquad -1<\alpha<0, \tag{A.24}$$

$$L(t,\alpha;\gamma) = -\frac{\alpha}{2}\ln\Big(-\widehat{X}^{11}(0)\big(\widehat{X}^{12}(0)\big)^{-1}s^{-\alpha}\Big)\Big|_{s=0}^{t}, \qquad \alpha>0. \tag{A.25}$$

The function $\widehat{\mathbf{X}}(\lambda)$ was introduced in RHP 6.2, condition (3).

Appendix B. Bessel and confluent hypergeometric parametrices

Following [15], section 6.1 we define for $\zeta \in \mathbb{C}\backslash[0,\infty)$ the unimodular matrix valued function

$$\Psi_\alpha(\zeta) = \sqrt{\pi}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\begin{bmatrix} I_\alpha((-\zeta)^{\frac{1}{2}}) & -\frac{\mathrm{i}}{\pi}K_\alpha((-\zeta)^{\frac{1}{2}}) \\ (-\zeta)^{\frac{1}{2}}I_\alpha'((-\zeta)^{\frac{1}{2}}) & -\frac{\mathrm{i}}{\pi}(-\zeta)^{\frac{1}{2}}K_\alpha'((-\zeta)^{\frac{1}{2}})\end{bmatrix}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\alpha\sigma_3} \tag{B.1}$$

using the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ both defined with their principal branches, cf. [60] and $z^\alpha : \mathbb{C}\backslash(-\infty,0] \to \mathbb{C}$ also in terms of the principal branch. Note that

$$\big(\Psi_\alpha(\zeta)\big)_+ = \big(\Psi_\alpha(\zeta)\big)_-\begin{bmatrix}\mathrm{e}^{-\mathrm{i}\pi\alpha} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha}\end{bmatrix}, \quad \zeta>0; \qquad \frac{\mathrm{d}\Psi_\alpha}{\mathrm{d}\zeta} = \begin{bmatrix} 0 & \frac{1}{2\zeta} \\ \frac{\alpha^2}{2\zeta} - \frac{1}{2} & 0\end{bmatrix}\Psi_\alpha.$$

and if we assemble the function (compare (6.3) in [15])

$$\Psi(\zeta;\alpha) = \Psi_\alpha(\zeta)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\alpha\sigma_3}\begin{cases}\left[\begin{smallmatrix}1 & 0 \\ -\mathrm{e}^{-\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg\zeta \in (0,\frac{\pi}{3}) \\ \mathbb{I}, & \arg\zeta\in(\frac{\pi}{3},\frac{5\pi}{3}) \\ \left[\begin{smallmatrix}1 & 0 \\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg\zeta\in(\frac{5\pi}{3},2\pi)\end{cases} \tag{B.2}$$

then

Riemann-Hilbert Problem B.1 (see [15], RHP 6.1.)**.** *The model function $\Psi(\zeta;\alpha)$ defined in (B.2) has the following properties*

(1) $\Psi(\zeta;\alpha)$ *is analytic for $\zeta\in\mathbb{C}\backslash(\Sigma_\Psi\cup\{0\})$ with $\Sigma_\Psi = \bigcup_{j=1}^3\Gamma_j$ where*

$$\Gamma_1 = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}(0,\infty), \qquad \Gamma_2 = (0,\infty), \qquad \Gamma_3 = \mathrm{e}^{\mathrm{i}\frac{5\pi}{3}}(0,\infty)$$

are all oriented from zero to infinity.

(2) *Along Σ_Ψ we observe the jumps*

$$\Psi_+(\zeta;\alpha) = \Psi_-(\zeta;\alpha)\begin{bmatrix}1 & 0 \\ \mathrm{e}^{-\mathrm{i}\pi\alpha} & 1\end{bmatrix}, \quad \zeta\in\Gamma_1; \qquad \Psi_+(\zeta;\alpha) = \Psi_-(\zeta;\alpha)\begin{bmatrix}1 & 0 \\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{bmatrix}, \quad \zeta\in\Gamma_3$$

and

$$\Psi_+(\zeta;\alpha) = \Psi_-(\zeta;\alpha)\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}, \quad \zeta\in\Gamma_2$$

(3) *In a vicinity of $\zeta=0$, first in case $\alpha\notin\mathbb{Z}$,*

$$\Psi(\zeta;\alpha)=\widehat{\Psi}(\zeta;\alpha)(-\zeta)^{\frac{\alpha}{2}\sigma_3}\begin{bmatrix}1 & \frac{\mathrm{i}}{2}\frac{1}{\sin\pi\alpha}\\ 0 & 1\end{bmatrix}\begin{cases}\left[\begin{smallmatrix}1 & 0\\ -\mathrm{e}^{-\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg\zeta\in(0,\frac{\pi}{3})\\ \mathbb{I}, & \arg\zeta\in(\frac{\pi}{3},\frac{5\pi}{3})\\ \left[\begin{smallmatrix}1 & 0\\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg\zeta\in(\frac{5\pi}{3},2\pi)\end{cases},$$

and second for $\alpha\in\mathbb{Z}$,

$$\Psi(\zeta;\alpha)=\widehat{\Psi}(\zeta;\alpha)(-\zeta)^{\frac{\alpha}{2}\sigma_3}\begin{bmatrix}1 & -\frac{\mathrm{e}^{\mathrm{i}\pi\alpha}}{2\pi\mathrm{i}}\ln(-\zeta)\\ 0 & 1\end{bmatrix}\begin{cases}\left[\begin{smallmatrix}1 & 0\\ -\mathrm{e}^{-\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg\zeta\in(0,\frac{\pi}{3})\\ \mathbb{I}, & \arg\zeta\in(\frac{\pi}{3},\frac{5\pi}{3})\\ \left[\begin{smallmatrix}1 & 0\\ \mathrm{e}^{\mathrm{i}\pi\alpha} & 1\end{smallmatrix}\right], & \arg\zeta\in(\frac{5\pi}{3},2\pi)\end{cases}.$$

In both cases $\widehat{\Psi}(\zeta;\alpha)$ is analytic at $\zeta=0$ and principal branches are chosen throughout.

Remark B.2. *With the help of* [60], *Chapter* 10.31 *we find*

$$\widehat{\Psi}(\zeta;\alpha)=\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\begin{bmatrix}\widehat{\Psi}^{11}(\zeta;\alpha) & -\frac{\mathrm{i}}{2}\frac{1}{\sin\pi\alpha}\widehat{\Psi}^{11}(\zeta;-\alpha)\\ \widehat{\Psi}^{21}(\zeta;\alpha) & -\frac{\mathrm{i}}{2}\frac{1}{\sin\pi\alpha}\widehat{\Psi}^{21}(\zeta;-\alpha)\end{bmatrix},\quad \alpha\notin\mathbb{Z}$$

with

$$\widehat{\Psi}^{11}(\zeta;\alpha)=\frac{1}{2^{\alpha}}\sum_{k=0}^{\infty}\frac{(-\frac{\zeta}{4})^k}{k!\,\Gamma(1+\alpha+k)},\quad \widehat{\Psi}^{21}(\zeta;\alpha)=\frac{\alpha}{2^{\alpha}}\sum_{k=0}^{\infty}\frac{(1+\frac{2k}{\alpha})(-\frac{\zeta}{4})^k}{k!\,\Gamma(1+\alpha+k)},\qquad \zeta\in\mathbb{D}_r(0);$$

and

$$\widehat{\Psi}(\zeta;\alpha)=\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left\{\begin{bmatrix}\frac{2^{-\alpha}}{\Gamma(1+\alpha)} & \frac{2^{\alpha}\Gamma(\alpha)}{2\pi\mathrm{i}}\\ \frac{\alpha 2^{-\alpha}}{\Gamma(1+\alpha)} & -\frac{\alpha 2^{\alpha}\Gamma(\alpha)}{2\pi\mathrm{i}}\end{bmatrix}+\mathcal{O}\big(\zeta\big)\right\},\quad \alpha\in\mathbb{Z}_{\geq 1},\qquad \zeta\in\mathbb{D}_r(0);$$

as well as

$$\widehat{\Psi}(\zeta;0)=\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left\{\begin{bmatrix}1 & \frac{\mathrm{i}}{\pi}(\gamma_E-\ln 2)\\ 0 & \frac{\mathrm{i}}{\pi}\end{bmatrix}+\mathcal{O}\big(\zeta\big)\right\},\qquad \zeta\in\mathbb{D}_r(0).$$

(4) *As $\zeta\to\infty,\zeta\notin\Sigma_\Psi$ we have*

$$\Psi(\zeta;\alpha)\sim\begin{bmatrix}1 & 0\\ -b_1(\alpha) & 1\end{bmatrix}\left\{\mathbb{I}+\sum_{m=1}^{\infty}\begin{bmatrix}a_{2m}(\alpha) & -a_{2m-1}(\alpha)\\ b_1(\alpha)a_{2m}(\alpha)-b_{2m+1}(\alpha) & b_{2m}(\alpha)-b_1(\alpha)a_{2m-1}(\alpha)\end{bmatrix}(-\zeta)^{-m}\right\}$$
$$\times(-\zeta)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sigma_3}\mathrm{e}^{(-\zeta)^{\frac{1}{2}}\sigma_3}$$

with the coefficients (cf. [60]*)*

$$a_k(\nu)=\frac{1}{k!\,8^k}(4\nu^2-1^2)(4\nu^2-3^2)\cdot\ldots\cdot(4\nu^2-(2k-1)^2),\ k\in\mathbb{Z}_{\geq 1};\qquad b_1(\nu)=\frac{1}{8}(4\nu^2+3)$$

and

$$b_k(\nu)=\frac{1}{k!\,8^k}\big((4\nu^2-1^2)(4\nu^2-3^2)\cdot\ldots\cdot(4\nu^2-(2k-3)^2)\big)(4\nu^2+4k^2-1),\quad k\in\mathbb{Z}_{\geq 2}.$$

In addition to the Bessel-parametrix (B.2) we require also the following model function built out of confluent hypergeometric functions $U(a,\zeta)\equiv U(a,1,\zeta)$. The underlying construction is essentially a rotation of the one given in [16], equations (2.19) and (2.21), see also [19, 46] for similar constructions. In more detail, define

$$\Phi_0(\zeta)=\begin{bmatrix}U(\nu,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta)\mathrm{e}^{2\pi\mathrm{i}\nu} & -U(1-\nu,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}}\zeta)\mathrm{e}^{\mathrm{i}\pi\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)}\\ -U(1+\nu,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta)\mathrm{e}^{\mathrm{i}\pi\nu}\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & U(-\nu,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}}\zeta)\end{bmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}\zeta\sigma_3},\quad \arg\zeta\in\left(\frac{\pi}{2},\frac{5\pi}{2}\right)\tag{B.3}$$

with $\nu = \frac{v}{2\pi\mathrm{i}} \in \mathrm{i}\mathbb{R}$. Now assemble

$$\Phi(\zeta) = \Phi_0(\zeta) \begin{cases} \mathbb{I}, & \arg\zeta \in (\pi, \frac{4\pi}{3}) \\ \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\mathrm{i}\pi\nu} & 1 \end{bmatrix}, & \arg\zeta \in (\frac{4\pi}{3}, \frac{3\pi}{2}) \\ \begin{bmatrix} 1 & 0 \\ 2\mathrm{i}\sin\pi\nu & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-\mathrm{i}\pi\nu} & 1 \end{bmatrix}, & \arg\zeta \in (\frac{3\pi}{2}, \frac{5\pi}{3}) \\ \begin{bmatrix} 1 & 0 \\ 2\mathrm{i}\sin\pi\nu & 1 \end{bmatrix}, & \arg\zeta \in (\frac{5\pi}{3}, 2\pi) \end{cases} \begin{cases} \begin{bmatrix} 1 & 0 \\ 2\mathrm{i}\sin\pi\nu & 1 \end{bmatrix}\begin{bmatrix} 0 & -\mathrm{e}^{\mathrm{i}\pi\nu} \\ \mathrm{e}^{-\mathrm{i}\pi\nu} & 0 \end{bmatrix}, & \arg\zeta \in (2\pi, \frac{7\pi}{3}) \\ \begin{bmatrix} 1 & 0 \\ 2\mathrm{i}\sin\pi\nu & 1 \end{bmatrix}\begin{bmatrix} 1 & -\mathrm{e}^{\mathrm{i}\pi\nu} \\ \mathrm{e}^{-\mathrm{i}\pi\nu} & 0 \end{bmatrix}, & \arg\zeta \in (\frac{7\pi}{3}, \frac{5\pi}{2}) \\ \begin{bmatrix} 1 & -\mathrm{e}^{-\mathrm{i}\pi\nu} \\ \mathrm{e}^{\mathrm{i}\pi\nu} & 0 \end{bmatrix}, & \arg\zeta \in (\frac{\pi}{2}, \frac{2\pi}{3}) \\ \begin{bmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\pi\nu} \\ \mathrm{e}^{\mathrm{i}\pi\nu} & 0 \end{bmatrix}, & \arg\zeta \in (\frac{2\pi}{3}, \pi) \end{cases} \tag{B.4}$$

and by recalling analytic and asymptotic properties of $U(a,\zeta)$, cf. [60] we find

Riemann-Hilbert Problem B.3. *The function $\Phi(\zeta) \in \mathbb{C}^{2\times 2}$ introduced in (B.3) and (B.4) has the following properties.*

(1) *$\Phi(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash(\{\arg\zeta = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}\} \cup \{0\})$ and we orient the six rays emanating from $\zeta = 0$ as shown near $\lambda = 1$ in Figure 13.*

(2) *The limiting values $\Phi_\pm(\zeta)$ on the jump contours satisfy*

$$\Phi_+(\zeta) = \Phi_-(\zeta)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}, \quad \arg\zeta = \frac{2\pi}{3}; \qquad \Phi_+(\zeta) = \Phi_-(\zeta)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}, \quad \arg\zeta = \pi;$$

$$\Phi_+(\zeta) = \Phi_-(\zeta)\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}, \quad \arg\zeta = \frac{4\pi}{3}; \qquad \Phi_+(\zeta) = \Phi_-(\zeta)\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}, \quad \arg\zeta = \frac{5\pi}{3};$$

$$\Phi_+(\zeta) = \Phi_-(\zeta)\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}, \quad \arg\zeta = 2\pi; \qquad \Phi_+(\zeta) = \Phi_-(\zeta)\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}, \quad \arg\zeta = \frac{7\pi}{3}.$$

By construction, there are no jumps on the vertical axis $\arg\zeta = \frac{\pi}{2}, \frac{3\pi}{2}$.

(3) *In a vicinity of $\zeta = 0$ we find*

$$\Psi(\zeta) = \widehat{\Psi}(\zeta)\begin{bmatrix} 1 & \frac{\gamma}{2\pi\mathrm{i}}\ln\zeta \\ 0 & 1 \end{bmatrix} \begin{cases} \begin{bmatrix} 1 & 0 \\ -\mathrm{e}^{2\pi\mathrm{i}\nu} & 1 \end{bmatrix}, & \arg\zeta \in (-\pi, -\frac{2\pi}{3}) \\ \mathbb{I}, & \arg\zeta \in (-\frac{2\pi}{3}, -\frac{\pi}{3}) \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, & \arg\zeta \in (-\frac{\pi}{3}, 0) \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, & \arg\zeta \in (0, \frac{\pi}{3}) \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, & \arg\zeta \in (\frac{\pi}{3}, \frac{2\pi}{3}) \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{2\pi\mathrm{i}\nu} & 1 \end{bmatrix}, & \arg\zeta \in (\frac{2\pi}{3}, \pi) \end{cases} \times \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}$$

with $\widehat{\Psi}(\zeta)$ analytic at $\zeta = 0$ and $\ln : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ defined with its principal branch.

(4) *As $\zeta \to \infty$,*

$$\Phi(\zeta) \sim \left\{\mathbb{I} + \sum_{m=1}^{\infty} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\begin{bmatrix} ((\nu)_m)^2 & (-1)^m\, m((1-\nu)_{m-1})^2\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ m\,((1+\nu)_{m-1})^2\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & (-1)^m((-\nu)_m)^2 \end{bmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu\sigma_3}\frac{(\mathrm{i}\zeta)^{-m}}{m!}\right\}$$

$$\times\ \zeta^{-\nu\sigma_3}\mathrm{e}^{\frac{\mathrm{i}}{2}\zeta\sigma_3}\begin{cases} \begin{bmatrix} 0 & -\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}\nu} \\ \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu} & 0 \end{bmatrix}, & \arg\zeta \in (\frac{\pi}{2}, \pi) \\ \begin{bmatrix} \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}\nu} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}\nu} \end{bmatrix}, & \arg\zeta \in (\pi, 2\pi) \\ \begin{bmatrix} 0 & -\mathrm{e}^{\mathrm{i}\frac{7\pi}{2}\nu} \\ \mathrm{e}^{-\mathrm{i}\frac{5\pi}{2}\nu} & 0 \end{bmatrix}, & \arg\zeta \in (2\pi, \frac{5\pi}{2}) \end{cases}$$

where $(a)_0 = 1, (a)_m = a(a+1)(a+2)\cdot\ldots\cdot(a+m-1), a \in \mathbb{C}, m \in \mathbb{Z}_{\geq 0}$ is the Pochhammer symbol.

Appendix C. Smoothness of $\mathcal{H}_H$

We first show the absence of singularities in q and p using probabilistic arguments. We know from the probabilistic interpretation of $F_H(t,\alpha)$ that $\mathcal{H}(q,p,t,\alpha)$ is smooth for $t \in (0,+\infty)$. But (1.13) implies that (see also (A.23) below)

$$t(q^2-1)(tq_t)_t = q(tq_t)^2 + \frac{1}{4}(t-\alpha^2)q + \frac{1}{4}tq^3(q^2-2).$$

Hence, near an assumed pole $t_0 > 0$ of $q(t,\alpha;\gamma)$, we have the Laurent expansion

$$q(t,\alpha;\gamma) = \frac{c}{t-t_0} + \mathcal{O}\big(t-t_0\big), \quad c^2 = 4t_0, \quad 0 < |t-t_0| < r,$$

and thus back in (1.13),

$$\mathcal{H}_H(q,p,t,\alpha)=\frac{1}{t-t_0}+\mathcal{O}(1),\qquad 0<|t-t_0|<r.$$

But this contradicts the regularity of $\mathcal{H}(q,p,t,\alpha)$, i.e. $q(t,\alpha;\gamma)$ is indeed smooth on $(0,+\infty)$ for $\alpha\in(-1,+\infty),\gamma\in[0,1]$. Alternatively we can start from Proposition A.1 and (A.17),

$$q^2(t,\alpha;\gamma)=-2\frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t}\ln\det\left(1-\gamma K_{\rm Bess}^{\alpha}\restriction_{L^2(0,t)}\right)\right), \tag{C.1}$$

so that possible poles of $q(t,\alpha;\gamma)$ coincide with zeros of the Fredholm determinant (as function of t). Thus, if we can estimate the operator norm of $\gamma K_{\rm Bess}^{\alpha}\restriction_{L^2(0,t)}$ for $t\in(0,+\infty)$ above by unity, regularity of q follows. In order to achieve this we draw inspiration from [65] and think of $K_{\rm Bess}^{\alpha}\restriction_{L^2(0,t)}$ as acting on $L^2(0,\infty)$ with kernel

$$\chi_{(0,t)}(\lambda)K_{\rm Bess}^{\alpha}(\lambda,\mu)\chi_{(0,t)}(\mu).$$

Hence, introducing the Hankel transform

$$H:L^2(0,\infty)\to L^2(0,\infty);\qquad (Hf)(\lambda)=\frac{1}{2}\int_0^{\infty}J_{\alpha}(\sqrt{\lambda\mu})f(\mu)\,\mathrm{d}\mu,$$

which is unitary on $L^2(0,\infty)$, cf. [51, 65], the operator $K_{\rm Bess}^{\alpha}\restriction_{L^2(0,t)}$ is equal to the square of P_tHP_t, where

$$P_t:L^2(0,\infty)\to L^2(0,t);\qquad (P_tf)(\lambda)=f(\lambda)\chi_{(0,t)}(\lambda)$$

is the orthogonal projection from $L^2(0,\infty)$ to $L^2(0,t)$ with $t>0$. This means that, for any fixed $t\in(0,+\infty)$,

$$\|P_tHP_t\|_{L^2(0,\infty)}\leq\|P_t\|^2_{L^2(0,\infty)}\cdot\|H\|_{L^2(0,\infty)}=1.$$

Now suppose that P_tHP_t were to have eigenvalues ±1 for some $t\in(0,\infty)$ with eigenfunctions $f_{\pm}\in L^2(0,\infty)$. Then $P_tf_{\pm}$ are also eigenfunctions to the same eigenvalues and

$$\langle HP_tf_{\pm},P_tf_{\pm}\rangle_{L^2(0,\infty)}=\langle P_tHP_tf_{\pm},P_tf_{\pm}\rangle_{L^2(0,\infty)}=\pm\|P_tf_{\pm}\|^2_{L^2(0,\infty)}. \tag{C.2}$$

But by Cauchy-Schwarz inequality

$$|\langle Hu,u\rangle_{L^2(0\infty)}|\leq\|Hu\|_{L^2(0,\infty)}\|u\|_{L^2(0,\infty)}=\|u\|^2_{L^2(0,\infty)},$$

with equality iff u is an eigenfunction. Thus, returning to (C.2), P_tf is a (nonzero) eigenfunction of the Hankel transform which vanishes for $\lambda>t$. But since $t\in(0,\infty)$ is fixed, $(HP_tf)(\lambda)$ is analytic for $\lambda\in\mathbb{C}\backslash(-\infty,0]$, i.e. we must have $P_tf=0$, which is a contradiction.

References

[1] M. Ablowitz, H. Segur, Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent, *Physica D*, **3**, 165-184 (1981).

[2] F.V. Andreev and A.V. Kitaev, Connection formulae for asymptotics of the fifth Painlevé transcendent on the real axis, *Nonlinearity*, **13** No. 5, 1801–1840 (2000).

[3] J. Baik, R. Buckingham, and J. DiFranco, Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function, *Commun. Math. Phys.* 280, 463-497 (2008).

[4] E. L. Basor, C. A. Tracy, Some problems associated with the asymptotics of τ-functions, *Surikagaku (Mathematical Sciences)* **30**, no. 3, 71–76 (1992).

[5] E. Basor and H. Widom, Toeplitz and Wiener-Hopf determinants with piecewise continuous symbols, *J. Funct. Anal.* **50**, 387-413 (1983).

[6] M. Bertola, The dependence on the monodromy data of the isomonodromic tau function, *Commun. Math. Phys.* **339**, 539-579 (2010).

[7] M. Bertola, Corrigendum: The dependence on the monodromy data of the isomonodromic tau function, preprint: arXiv:1601.04790.

[8] A. Bogatskiy, T. Claeys, and A. Its, Hankel determinant and orthogonal polynomials for a Gaussian weight with a discontinuity at the edge, *Commun. Math. Phys.* **347**, 127-162 (2016).

[9] O. Bohigas, M. Pato, Deformations of the Tracy-Widom distribution function, *Phys. Rev. E* **79**, 031117 (2009).

[10] O. Bohigas and M. Pato, Missing levels in correlated spectra, *Phys. Lett.* B 595, 171-176 (2004).

[11] O. Bohigas and M. Pato, Randomly incomplete spectra and intermediate statistics, *Phys. Rev.* E 74, 036212 (2006).

[12] A. Bolibruch, A. Its, A. Kapaev, On the Riemann-Hilbert-Birkhoff inverse monodromy problem and the Painlevé equations, *Algebra Anal.* **16**, 121-162 (2004).

[13] F. Bornemann, On the numerical evaluation of Fredholm determinants, *Mathematics of Computation*, Vol. 79, Number 270, 871-915 (2009).

[14] T. Bothner, Transition asymptotics for the Painlevé II transcendent, *Duke Mathematical Journal* **166**, 205-324 (2017).
[15] T. Bothner, From gap-asymptotics to eigenvalue expansions, *J. Phys. A: Mathematical and Theoretical* **49** 075204 (2016).
[16] T. Bothner, R. Buckingham, Large deformations of the Tracy-Widom distribution I. Non-oscillatory asymptotics, *Commun. Math. Phys.* **359**, 223-263 (2018).
[17] T. Bothner, P. Deift, A. Its, I. Krasovsky, On the asymptotic behavior of a log gas in the bulk scaling limit in the presence of a varying external potential II, *Operator Theory: Advances and Applications*, Vol. 259, 213-234 (2017).
[18] T. Bothner, P. Deift, A. Its, I. Krasovsky, The sine process under the influence of a varying potential, preprint: arXiv:1807.11387
[19] T. Bothner, A. Its, Asymptotics of a cubic sine kernel determinant, *St. Petersburg Math. J.* **26** (2014), 22-92.
[20] A. M. Budylin, V. S. Buslaev, Quasiclassical asymptotics of the resolvent of an integral convolution operator with a sine kernel on a finite interval, (Russian) *Algebra i Analiz* **7**, no. 6, (1995), 79–103; translation in *St. Petersburg Math. J.* **7**, no. 6 (1996), 925–942.
[21] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics. New York, NY, (1999).
[22] P. Deift, A. Its, and I. Krasovsky, Asymptotics of the Airy-kernel determinant, *Commun. Math. Phys.* **278**, 643-678 (2008).
[23] P. Deift, A. Its, I. Krasovsky, Z. Zhou, The Widom-Dyson constant for the gap probability in random matrix theory, *J. Comput. Appl. Math.* **202**, no. 1, 26-47 (2007).
[24] P. Deift, A. Its, X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, *Ann. of Math.* **146** (1997), 149-235.
[25] P. Deift, I. Krasovsky, J. Vasilevska, Asymptotics for a determinant with a confluent hypergeometric kernel, *International Mathematics Research Notices*, Vol 2001, No. **9**, 2117-2160 (2010).
[26] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. of Math.* **137**, 296-368 (1993).
[27] P. Deift, X. Zhou, Asymptotics for the Painlevé II equation, *Comm. Pure Appl. Math.* **48**, 277-337 (1995).
[28] F. Dyson, Fredholm determinants and inverse scattering problems, *Commun. Math. Phys.* **47**, 171-183 (1976).
[29] T. Ehrhardt, Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel, *Comm. Math. Phys.* **262** 317-341 (2006).
[30] T. Ehrhardt, The asymptotics of a Bessel-kernel determinant which arises in Random Matrix Theory, *Advances in Mathematics* **225**, 3088-3133 (2010).
[31] H. Flaschka and A.C. Newell, Monodromy- and spectrum-preserving deformations I, *Comm. Math. Phys.* **76** (1980) 65–116.
[32] A. Fokas, A. Its, A. Kapaev, V. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert Approach*, AMS Series: Mathematical Surveys and Monographs, vol. 128, 2006.
[33] P. Forrester, The spectrum edge of random matrix ensembles, *Nuclear Physics* **B402**, 709-728 (1993).
[34] P. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.
[35] P. Forrester, N. Witte, Application of the τ-function theory of Painlevé equations to random matrices: PV, PIII, the LUE, JUE and CUE, *Commun. Pure Appl. Math.* **55**, 679-727 (2002).
[36] P. Forrester, N. Witte, τ-function evaluation of gap probabilities in orthogonal and symplectic matrix ensembles, *Nonlinearity* **15**, 937-954 (2002).
[37] P. Gavrylenko, O. Lisovyy, *Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions*, *Commun. Math. Phys.* **363**, 1-58 (2018).
[38] O. Gamayun, N. Iorgov, O. Lisovyy, Conformal field theory of Painlevé VI, *JHEP* **10**, (2012).
[39] O. Gamayun, N. Iorgov, O. Lisovyy, How instanton combinatorics solves Painlevé VI, V and III's, *J. Phys.* **A46**, (2013), 335203.
[40] S. Hastings, J. McLeod, A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation, *Arch. Ration. Mech. Anal.* **73**, 31-51 (1980).
[41] E. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.
[42] N. Iorgov, O. Lisovyy, Yu. Tykhyy, Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks, *JHEP* **12**, (2013), 029.
[43] A. Its, O. Lisovyy, Yu. Tykhyy, Connection problem for the sine-Gordon/Painlevé III tau function and irregular conformal blocks, *Int. Math. Res. Not.,* (2014), rnu209.
[44] N. Iorgov, O. Lisovyy, J. Teschner, Isomonodromic tau-functions from Liouville conformal blocks, *Comm. Math. Phys.* **336**, (2015), 671–694.
[45] A. Its, A. Izergin, V. Korepin, N. Slavnov, Differential equations for quantum correlation functions, *Int. J. Mod. Phys. B* **4**, 1003-1037 (1990).
[46] A. Its, I. Krasovsky, Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump, *Contemp. Math.* **458**, 215-247 (2008).
[47] A. Its, O. Lisovyy, A. Prokhorov, Monodromy dependence and connection formulæ for isomondromic tau functions, *Duke Math. J.* Volume 167, Number 7 (2018), 1347-1432.
[48] A. Its, A. Prokhorov, Connection problem for the tau-function of the Sine-Gordon reduction of Painlevé-III equation via the Riemann-Hilbert approach, *Int. Math. Res. Notices* **22**, 6856-6883 (2016).

[49] M. Jimbo, T. Miwa, Y. Mori, M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, *Physica* **1D** (1980) 80-158.
[50] M. Jimbo, T. Miwa, K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I, *Physica* **D2**, 306-352 (1981).
[51] B. Kostant, On Laguerre polynomials, Bessel functions, Hankel transform and a series in the unitary dual of the simply-connected covering group of $SL(2,\mathbb{R})$, *Representation theory*, Volume 4, Pages 181-224 (2000).
[52] I. Krasovsky, Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle, *Int. Math. Res. Not.* 2004, 1249-1272 (2004).
[53] I. Krasovsky, Aspects of Toeplitz determinants, *Progress in Probability*, Vol. 64, 305-324 (2011).
[54] O. Lisovyy, Dyson's constant for the hypergeometric kernel, in "New trends in quantum integrable systems" (eds. B. Feigin, M. Jimbo, M. Okado), World Scientific, (2011), 243–267.
[55] O. Lisovyy, J. Roussillon, On the connection problem for Painlevé I, *J. Phys. A: Math. Theor.* **50** (2017), no. 25, 255202, 15 pp.
[56] B. Malgrange, Sur les déformations isomonodromiques, I. Singularités régulieres, in *Mathematics and Physics*, Prog. Math. **37**, Birkhäuser, Boston, MA, 401-426 (1983).
[57] M. L. Mehta, *Random matrices*, San Diego, Academic Press, second edition, 1991.
[58] B.M. McCoy and Sh. Tang, Connection formulae for Painlevé V functions, *Physica D* v.19, 42-72, 1986.
[59] B.M. McCoy and Sh. Tang, Connection formulae for Painlevé V functions: II. The δ-function Bose gas problem, *Physica D* v. 20, 187-216, 1986.
[60] NIST Digital Library of Mathematical Functions, http://dlmf.nist.gov.
[61] J. Palmer, Zeros of the Jimbo, Miwa, Ueno tau function, *J.Math.Phys.* **40**(12),6638-6681(1999).
[62] B. Suleimanov, On asymptotics of regular solutions for a special kind of Painlevé V equation, Lect. Notes in Math., Springer Verlag, **1191**, 230-260 (1986).
[63] C. A. Tracy, Asymptotics of the τ-function arising in the two-dimensional Ising model, *Comm. Math. Phys.* **142**, 297–311 (1991).
[64] C. Tracy, H. Widom, Level-spacing distributions and the Airy kernel, *Commun. Math. Phys.* **159**, 151-174 (1994).
[65] C. Tracy, H. Widom, Level spacing distributions and the Bessel kernel, *Commun. Math. Phys.* **161**, 289-309 (1994).
[66] H. Widom, Analysis for the Fredholm determinant of the sine kernel on a union of intervals, *Commun. Math. Phys.* **171**, 159-180 (1995).

Department of Mathematics, King's College London, Strand, London, WC2R 2LS, United Kingdom
E-mail address: `thomas.bothner@kcl.ac.uk`

Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 N. Blackford St., Indianapolis, IN 46202, U.S.A. and St. Petersburg State University, Universitetskaya emb. 7/9, 199034, St. Petersburg, Russia
E-mail address: `aits@iupui.edu`

Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 N. Blackford St., Indianapolis, IN 46202, U.S.A. and St. Petersburg State University, Universitetskaya emb. 7/9, 199034, St. Petersburg, Russia
E-mail address: `aprokhor@iupui.edu`